Arithmetic Series

The first amphitheatres were built for contests between gladiators. Modern amphitheatres are usually used for the performing arts. Amphitheatres generally get wider as the distance from the stage increases.

Suppose a small amphitheatre can seat 15 people in the first row and each row can seat 3 more people than the previous row. The numbers of seats in the rows of this amphitheatre form the arithmetic *sequence* 15, 18, 21, 24, … If we wanted to determine the number of people who could sit in the first four rows, we would add the first four terms of the sequence. The sum 15 + 18 + 21 + 24 is referred to as an arithmetic *series.*

Arithmetic Series – The sum of the terms of an arithmetic sequence

To develop a formula for the sum of an arithmetic series, consider the following example:

Find the sum of the first 10 odd natural numbers.
This sum can be written as: $S_n = 1+3+5+7+9+11+13+15+17+19$

One way to write the sum:	$S = 1+3+5+7+9+11+13+15+17+19$
You can also write this in reverse order:	$S = 19+17+15+13+11+9+7+5+3+1$
Add the two equations:	$2S = 20+20+20+20+20+20+20+20+20+20$
Simplify:	$2S = (10)20$

$$\frac{2S}{2} = \frac{(10)20}{2}$$
$$S = \frac{200}{2}$$
$$S = 100$$

In general:

$$S_n = t_1 + t_2 + t_3 + t_4 + \ldots + t_n$$
$$S_n = t_1 + (t_1 + d) + (t_1 + 2d) + (t_1 + 3d) + \ldots + (t_1 + (n-1)d) \rightarrow \textit{equation } 1$$

In reverse order :

$$S_n = t_n + t_{n-1} + t_{n-2} + t_{n-3} + \ldots + t_1$$
$$S_n = t_n + (t_n - d) + (t_n - 2d) + (t_n - 3d) + \ldots + (t_n - (n-1)d) \rightarrow \textit{equation } 2$$

Add equations 1 *and* 2 *and solve for* S_n :

$$2S_n = (t_1 + t_n) + (t_1 + t_n) + (t_1 + t_n) + (t_1 + t_n) + \ldots + (t_1 + t_n)$$
$$2S_n = n(t_1 + t_n)$$
$$S_n = \frac{n}{2}(t_1 + t_n)$$
$$= \frac{n}{2}(t_1 + t_1 + d(n-1))$$
$$= \frac{n}{2}(2t_1 + d(n-1))$$

FORMULAS FOR THE SUM OF AN ARITHMETIC SERIES

$S_n = \frac{n}{2}(t_1 + t_n)$	OR $S_n = \frac{n}{2}(2t_1 + d(n-1))$
S_n = sum of the first n terms	S_n = sum of the first n terms
n = number of terms	n = number of terms
t_1 = first term	t_1 = first term
t_n = n^{th} term	d = common difference

Note: S_n is often called an n^{th} *partial sum* since it can represent the sum of a certain "part" of a sequence.

Example 1: Determine the Sum of an Arithmetic Series

Male fireflies flash in various patterns to signal locations or to ward off predators. Different species of fireflies have different flash characteristics, such as the intensity of the flash and the shape of the flash. Suppose that, under certain circumstances, a particular firefly flashes twice in the first minute, four times in the second minute, and six times in the third minute.

a. If this arithmetic pattern continues, what is the number of flashes in the 30^{th} minute?

b. What is the total number of flashes in 30 minutes?

Solution:

a. Number of flashes in the 30^{th} minute:

b. Total number of flashes in 30 minutes:

Example 2: Determine the Number of Terms in an Arithmetic Series

The sum of $(-8)+(-2)+(4)+\ldots+t_n$ is 1600. Determine the number of terms in the series.

Solution: t_1 = ________ d = __________ S_n = _________

$$S_n = \frac{n}{2}\left(2t_1 + d(n-1)\right)$$

Example 3: Determine the Terms of an Arithmetic Series

The sum of the first six terms of an arithmetic series is 99 and the sum of the first ten terms is 265.

a. Determine the first six terms of the series.
b. Calculate the sum of the first fifteen terms.

Solution:

a. Determine the first six terms of the series.

b. Calculate the sum of the first fifteen terms.

Sigma Notation

Sigma, or summation, notation is a shorthand notation used to represent sums with more than a few terms. This shorthand notation uses the Greek letter, Σ, which denotes a sum.

Example 4: Writing a Sum from Sigma Notation

Write the terms of the following sum:

$$\sum_{n=1}^{8}(3n+4) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Example 5: Using Sigma Notation to Represent a Sum

a. Use sigma notation to represent the sum of the arithmetic series: –5 + –1 + 3 + 7 + 11 + 15

Solution:

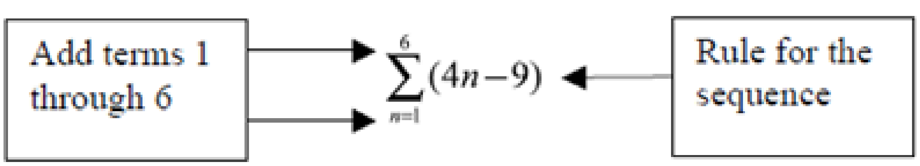
There are ______ terms in this series.

The formula that can be used to generate the terms of this series is:

$t_n = t_1 + d(n-1)$

The series –5 + –1 + 3 + 7 + 11 + 15 can be expressed using sigma notation as $\sum_{n=1}^{6}(4n-9)$.

This means:



It is important to note that the use of parentheses is required since $\sum_{n=1}^{6}4n-9$ would yield a different sum:

$\sum_{n=1}^{6}(4n-9) = -5+-1+3+7+11+15 = 30$ * $\sum_{n=1}^{6}4n-9 = 4+8+12+16+20+24-9 = 75$

* If the brackets are not included, the –9 is added at the *end* of the series.

b. Use sigma notation to represent the sum of the arithmetic series: –18 + –11 + –4 + 3 + … + 129

Example 6: Understanding Sigma Notation

Find the sum $\sum_{k=3}^{7}(2k+5)$.

Method 1	Method 2
Find *each* term in the indicated series and then add the terms. $\sum_{k=3}^{7}(2k+5)$	Use one of the arithmetic series formulas. $\sum_{k=3}^{7}(2k+5)$

Extra Practice

1. Write each of the following arithmetic series in expanded form and find the sum using two different methods.

   a. $\sum_{j=2}^{11}(-3j+2)$

b. $\sum_{n=4}^{12}(5n-3)$

2. If $\sum_{k=3}^{x}(6k-5)=928$, determine the value of x.

3. Express the arithmetic series (–16) + (–19) + (–22) + … + (–37) using sigma notation.