Infinite Geometric Series

With their opponent on the 10-yard line, the defense is penalized half the distance to the goal, placing the ball on the 5-yard line. If the defense continues to be penalized in this way, will the ball eventually reach the goal line? How many total penalty yards will the defense have incurred? This situation can be modeled by the infinite geometric series 5 + 2.5 + 1.25 + …

Infinite geometric series – A geometric series that has an infinite number of terms

Convergent Series – An infinite series for which the sequence of partial sums approaches a finite limit

For example, the sequence of partial sums of the infinite geometric series 0.9 + 0.09 + 0.009 + 0.0009 + … is 0.9, 0.99, 0.999, 0.9999, … This sequence approaches a finite limit of 1, so the series 0.9 + 0.09 + 0.009 + 0.0009 + … is *convergent* (and has a finite sum equal to 1). Note that r = ______ in this example. An infinite geometric series is convergent for $-1 < r < 1$.

Divergent Series – A series that does not converge

For example, the sequence of partial sums of the infinite geometric series 1 + 2 + 4 + 8 + … is 1, 3, 7, 15, … This sequence does not converge to a finite limit, so the series 1 + 2 + 4 + 8 + … is *divergent*. Note that r = ______ in this example. An infinite geometric series is divergent for $r > 1$ or $r < -1$.

To develop a formula for the sum of an *infinite* geometric series, consider the following:

When calculating the sum of a *finite* geometric series, we used the formula: $S_n = \dfrac{t_1 r^n - t_1}{r-1}$

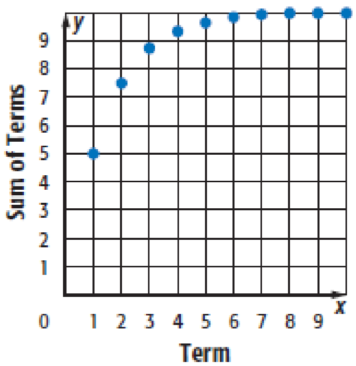
- For an *infinite* geometric series, if $-1 < r < 1$, the series will converge, and a *finite sum* exists. Note that in the above formula, the term $t_1 r^n$ will approach zero as n approaches infinity (since repeatedly multiplying a proper fraction by itself decreases its value toward zero). In this case, the term $t_1 r^n$ can be "ignored", and the formula becomes: $S_\infty = \dfrac{-t_1}{r-1} = \dfrac{t_1}{1-r}$
- Note that for an infinite geometric series, if $r > 1$ or $r < -1$, then the term $t_1 r^n$ will never approach a limit, so a sum *does not exist*.

FORMULA FOR THE SUM OF AN INFINITE GEOMETRIC SERIES

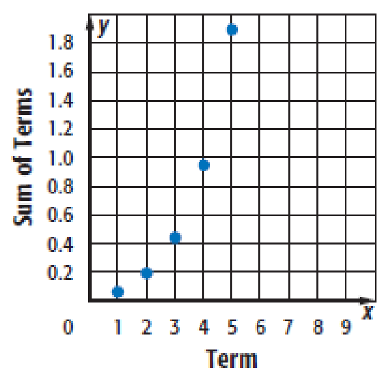
$S_\infty = \dfrac{t_1}{1-r}$, where $-1 < r < 1$ $\quad S_\infty$ = sum of an infinite number of terms $\quad t_1$ = first term $\quad r$ = common ratio

Example 1: Determine the Sum of an Infinite Geometric Series

In the football problem, the total number of penalty yards can be represented by the infinite geometric series $5+2.5+1.25+\dots$. Calculate this sum.



The graph of S_n for $1 \le n \le 10$ is shown to the left.
Notice that as n increases, S_n approaches 10.



For comparison, the graph of S_n for $1 \le n \le 5$ for the infinite geometric series $\frac{1}{16}+\frac{1}{8}+\frac{1}{4}+\dots$, where r = 2, is shown to the right. Notice that as n increases, S_n *does not* approach a fixed value, therefore the series *does not* have a sum.

Example 2: Sum of an Infinite Geometric Series

Decide whether each infinite geometric series is convergent or divergent. State the sum of the series, if it exists.

a. $\frac{1}{3}-1+3-9+\dots$

b. $12+3+\frac{3}{4}+\frac{3}{16}+\dots$

Solution:

a. $\frac{1}{3}-1+3-9+\dots$

b. $12+3+\frac{3}{4}+\frac{3}{16}+\dots$

Example 3: Determine the First Term of an Infinite Geometric Series

The sum of an infinite geometric series is 92 and the common ratio is $\frac{1}{4}$. What is the value of the first term?

Solution:

$$S_\infty = \frac{t_1}{1-r}$$

Example 4: Apply the Sum of an Infinite Geometric Series to Express a Repeating Decimal as a Fraction

Express the following repeating decimals as fractions.

a. $0.\overline{36}$ b. $4.1\overline{6}$

Solution:

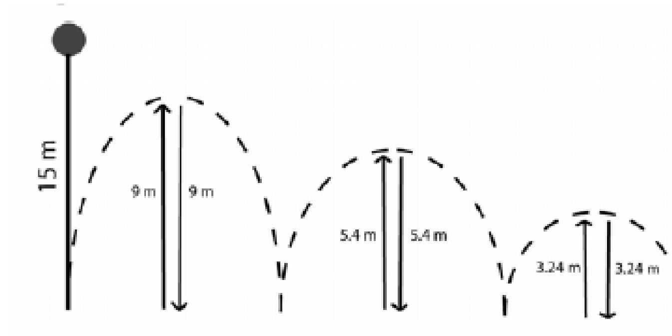
a. $0.\overline{36}$ b. $4.1\overline{6}$

Example 5: Apply the Sum of an Infinite Geometric Series

A ball is dropped from a height of 15 m and bounces to a height of 9 m. The ball continues to bounce to 60% of the previous height.

a. How far has the ball traveled vertically by the time it hits the ground for the seventh time?

b. How far has the ball traveled vertically by the time it comes to rest?



Solution:

a. By the time the ball hits the ground for the seventh time, it has traveled a total vertical distance of:

$[2(15+ 9 + 5.4 + 3.24 + \ldots + t_7) - 15]$ m

So, first determine the sum of the series $15+ 9 + 5.4 + 3.24 + \ldots + t_7$

$t_1 =$ _________ $r =$ _________ $n =$ _________

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$

Then calculate the total distance:

Total distance = ____

b. Determine the sum of the infinite series $15+ 9 + 5.4 + 3.24 + \ldots$

$$S_\infty = \frac{t_1}{1 - r}$$

Then calculate the total distance:

Total distance = ____