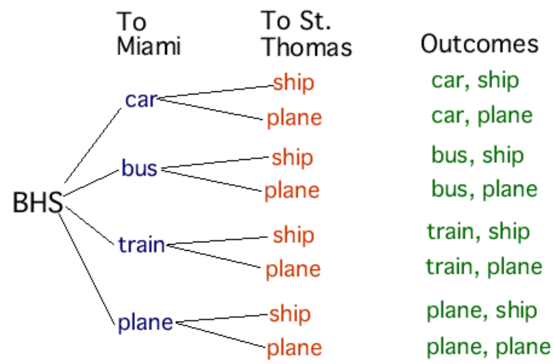
Permutations

Many questions in mathematics involve *counting*. For example, in how many ways can a committee of two men and three women be chosen from a group of 35 men and 40 women? How many different license plates can be made using three letters followed by three digits? How many different poker hands are possible?

Counting methods are used to determine the number of outcomes of an event. A tree diagram, for example, is one such method. A *tree diagram* is a graphic organizer used to list all possibilities of a sequence of events in a systematic way.

Example:

The senior class at BHS is planning a trip to the Virgin Islands. The class officers are exploring various travel options. From the high school, they can travel to Miami either by car, bus, train or plane. To travel to St. Thomas from Miami, they can take a plane or a ship. A tree diagram clearly illustrates each of the 8 possible ways the group can travel on their trip.



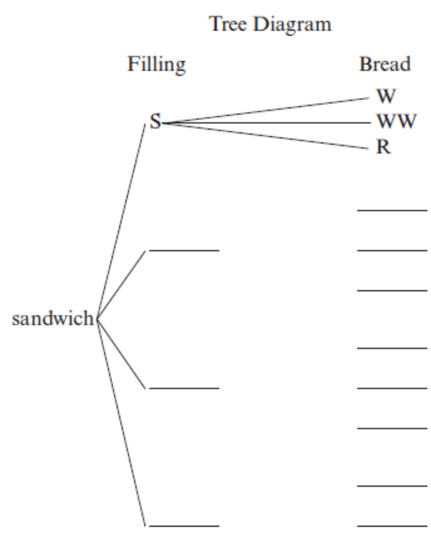
Example 1: Arrangements without Restrictions

A school cafeteria offers sandwiches made with fillings of salami (S), ham (H), cheese (C), or egg (E), all of which are available on white (W), whole wheat (WW) or rye (R) bread. How many different sandwiches can be made using only one filling?

Solution:

Method 1: List Outcomes and Count the Total

Use a tree diagram, or list all of the sandwich choices in a table, and count the outcomes.



List

Filling	Bread
S	W

______ possibilites

Method 2: Fundamental Counting Principle

______________________ x ______________________ = ______________________

(number of choices for filling) (number of choices for bread) (number of different sandwiches)

FUNDAMENTAL COUNTING PRINCIPLE

If one task can be performed in *a* ways and a second task can be performed in *b* ways, then the two tasks can be performed in *a x b* ways.

Example 2: Arrangements with Restrictions

In how many ways can five black (B) cars and four red (R) cars be parked next to each other in a parking garage if a black car has to be first and a red car has to be last?

Solution:

Whenever you encounter a situation with constraints or restrictions, always address the choices for the restricted positions first.

Use 9 blanks to represent the nine cars parked in a row. A black car must be in the first position and a red car must be in the last position. Fill these positions first.

_____ _____ _____ _____ _____ _____ _____ _____ _____

There are ______ black cars for the first position. There are ______ red cars for the last position.
After filling the end positions, there are ______ positions to fill with ______ cars remaining.
Use these numbers to fill in the blanks that represent the nine cars parked in a row.

By the fundamental counting principle, there are

(______)(______)(______)(______)(______)(______)(______)(______)(______)

= _________________ ways to park the cars in a row.

FACTORIAL NOTATION

Suppose you have 5 slips of paper, each containing a different test question. You are asked to make a test with 5 questions. In how many ways can you arrange the questions?

By the fundamental counting principle, there are $5\times4\times3\times2\times1=120$ ways to arrange the questions.

The shorthand way to express this product is 5!

Factorial: For any positive integer, n, the product of all the positive integers up to and including n.

In general, n! = (n)(n–1)(n–2)(n–3)(n–4)….(3)(2)(1)

For example, 8! = (8)(7)(6)(5)(4)(3)(2)(1)

By definition, 0! = 1

Example 3: Using Factorial Notation

a. Simplify and evaluate:

i. $\dfrac{8!}{5!} =$

ii. $\dfrac{10!}{(10-4)!} =$

iii. $\dfrac{6!}{2!(6-2)!} =$

iv. $\dfrac{n!}{(n-2)!} =$

b. Show that $20! - 19! + 18! = (362)(18!)$

PERMUTATIONS

A permutation is an arrangement in which a number of quantities are chosen with attention being paid to the order of choice. Examples of permutations include batting order for baseball team, license plate numbers, finish order of a race, PIN's for a bank account.

Example 4: Formula for Permutations

Suppose there are eight runners in the final of a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals if all possible outcomes of the race can occur and there are no ties?

Solution:

Using the fundamental counting principle, there are (_______)(_______)(_______) = _________ possible ways to award the medals. This product can be written using factorials:

$$8 \times 7 \times 6 = \frac{8!}{5!} = \frac{8!}{(8-3)!} = {}_8P_3$$

The notation ${}_nP_r$ is used to represent the number of permutations, or arrangements in a definite order, of r items taken from a set of n distinct items.

Formula for Permutations: ${}_nP_r = \dfrac{n!}{(n-r)!},\ n \in N$

Example 5: Working with Permutations

How many ways can you arrange all of the letters in the word MATH?

Solution:

Method 1: Fundamental Counting Principal

Method 2: Formula for Permutations

Example 6: Working with Permutations

a. How many 3-digit numbers can be created from the digits 1 to 7, inclusive, without repeating any digits?

b. If there are 40 clarinet players competing for places in the New Brunswick Youth Orchestra, how many ways can the 1st and 2nd chairs be filled?

Example 7: Permutations with Repeating Objects

How many *distinct* ways can you arrange all of the letters in the "word" SHHH?

If we proceed as we did in the MATH problem (Example 5) above, we would get a total of 4! possibilities. Unfortunately, this method over-counts since three of the letters are the same, so we need to correct for this. But how?

Let's pretend that the three H's are different, and call them H_1, H_2, and H_3. The 4! = 24 arrangements of the letters in SHHH would look like this:

$$SH_1H_2H_3,\ SH_1H_3H_2,\ SH_2H_1H_3,\ SH_2H_3H_1,\ SH_3H_1H_2,\ SH_3H_2H_1 \ \Rightarrow\ SHHH$$
$$H_1SH_2H_3,\ H_1SH_3H_2,\ H_2SH_1H_3,\ H_2SH_3H_1,\ H_3SH_1H_2,\ H_3SH_2H_1 \ \Rightarrow\ HSHH$$
$$H_1H_2SH_3,\ H_1H_3SH_2,\ H_2H_1SH_3,\ H_2H_3SH_1,\ H_3H_1SH_2,\ H_3H_2SH_1 \ \Rightarrow\ HHSH$$
$$H_1H_2H_3S,\ H_1H_3H_2S,\ H_2H_1H_3S,\ H_2H_3H_1S,\ H_3H_1H_2S,\ H_3H_2H_1S \ \Rightarrow\ HHHS$$

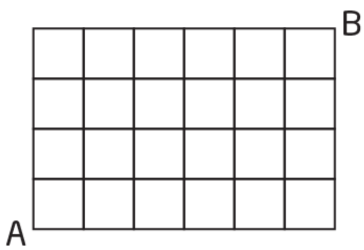
The 24 arrangements are really just 4 *distinct* arrangements.

So the number of distinct arrangements of the letters in the word SHHH is $\frac{4!}{3!} = \frac{24}{6} = 4$

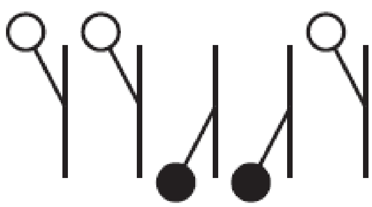
Example 8: Permutations with Repeating Objects

a. How many different twelve-letter arrangements can you make using the letters of NEWFOUNDLAND?

b. How many paths can you follow from A to B in a four by six rectangular grid if you move only up or to the right?



c. An electrical panel has five switches. In how many ways can the switches be positioned if three switches must be up and two must be down?



Example 9: Permutations where Repetitions are Allowed

A phone number in British Columbia consists of one of four area codes (236, 250, 604, or 778), followed by a 7-digit number that cannot begin with a 0 or 1. How many unique phone numbers are there?

Solution:

Example 10: Permutations with Constraints

Angie, Chelsea, Emily, Hamid, John, and Leah are seated in the front row in Mrs. Scholten's classroom. In how many ways can they be arranged if

a. Emily must be seated at the third desk?

b. Chelsea can't be at either end of the row?

c. The row starts with exactly two females?

d. Angie, Hamid, and John must be seated together?

e. Leah and John cannot be seated together?

Example 11: Permutations with Constraints

How many arrangements of all the letters in the word DAUGHTER are there if none of the vowels can ever be together?

Solution:

ARRANGEMENTS REQUIRING CASES

To solve some problems, you must count the different arrangements in all the cases that together cover all the possibilities. Calculate the number of arrangements for each case and then *add* the values for all cases to obtain the total number of arrangements.

Example 12: Using Cases to Determine Permutations

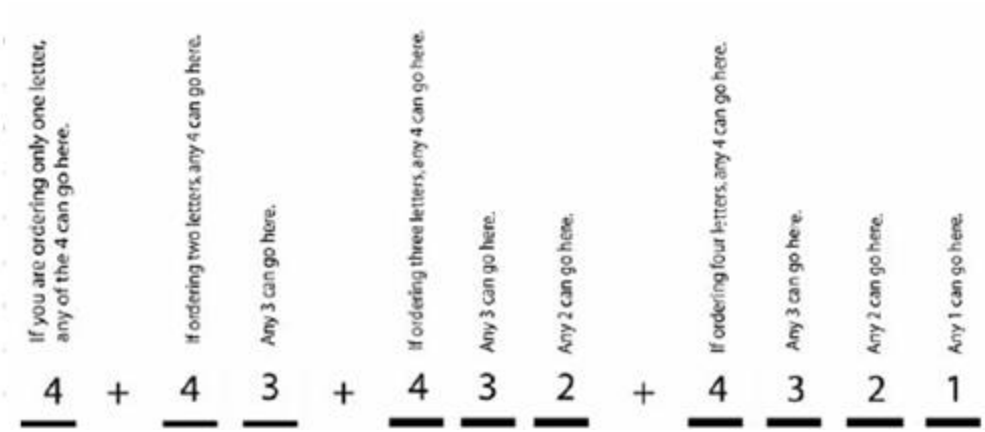
a. How many "words" (of any number of letters) can be formed from the letters C A N S?
b. How many "words" (with at least four letters) can be formed from the letters S U N D A Y?

Solution:

a. How many "words" (of any number of letters) can be formed from the letters C A N S?

There are four cases to be considered; 1 letter words, 2 letter words, 3 letter words, and 4 letter words:

Case 1 **Case 2** **Case 3** **Case 4**



We could also write this using permutations: ________ + ________ + ________ + ________ = ________

b. How many "words" (with at least four letters) can be formed from the letters S U N D A Y?

Example 13: Using Cases to Determine Permutations

How many four-digit positive integers less than 4670 can be formed using only the digits 1, 3, 4, 5, 8, 9 if repetition of digits is *not* allowed?

Solution:

Case 1: Numbers in the 4000's

There is _____ possibility for the first digit. The second digit has _____ possibilities. There are _____ possibilities for the next digit since any remaining number can be used, and _____ possibilities for the last digit.

Number of possibilities = ______________________________

Case 2: Numbers in the 1000's and 3000's

There are _____ possibilities for the first digit. Then the remaining digits would have _____ possibilities, then _____, and finally _____.

Number of possibilities = ______________________________

Total number of possibilities = _______________

Example 14: Using Cases to Determine Permutations

In how many ways can eight basketball players sit on a bench if either the one centre or both of the two forwards must sit at the end where the coach always sits?

Solution:

Case 1: The centre sits beside the coach

Case 2: The two forwards sit beside the coach

Total number of seating arrangements = ____________

Example 15: Solving Equations with Factorials & Permutations

Solve for n.

a. $\frac{n!}{10} = {}_{n-1}P_{n-3}$

b. ${}_{n+3}P_2 = 20$