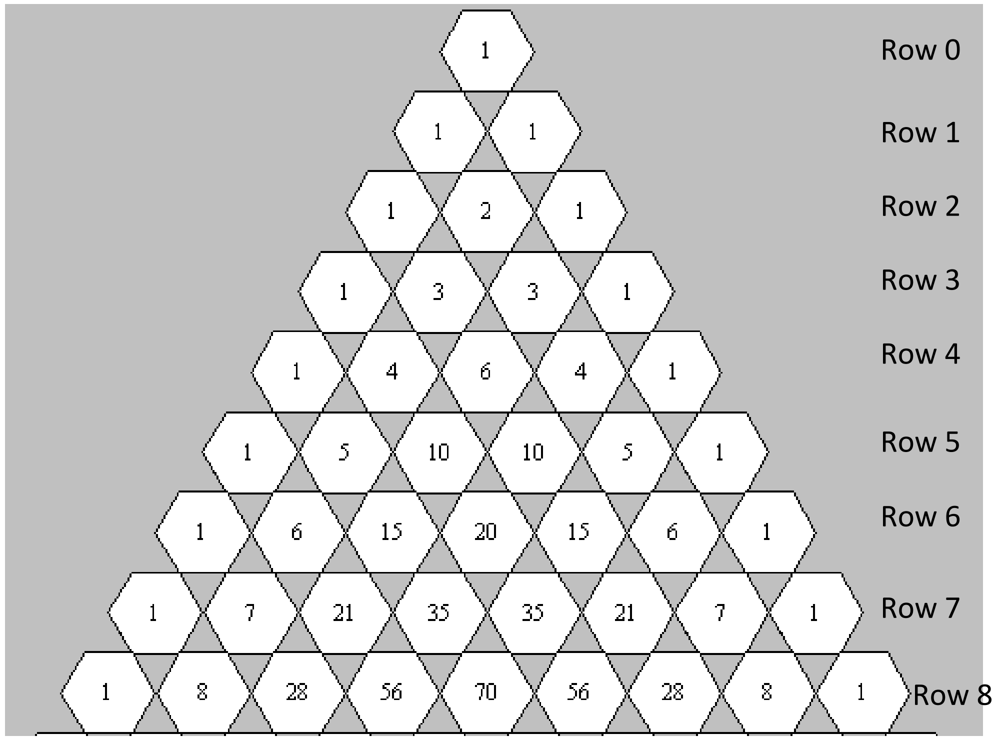
The Binomial Theorem



PASCAL'S TRIANGLE

Pascal's triangle is a triangular arrangement of numbers constructed in such a way that every number in the interior of the triangle is the sum of the two numbers directly above it. It has many applications, including determining the coefficients of the expansion of a binomial expression or finding the number of ways to choose m objects from a group of n indistinguishable objects.

Investigate Pascal's Triangle:

1. Examine Pascal's Triangle. Identify at least 3 patterns.

2. One row of Pascal's triangle is

   1 12 66 220 495 792 924 792 495 220 66 12 1

   a. What row would this be in Pascal's triangle?

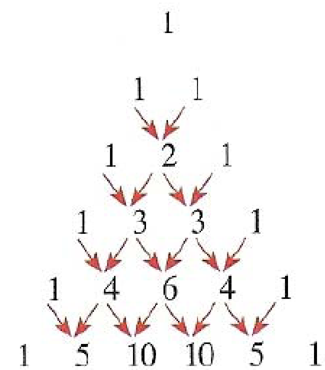
   b. Determine the next row in the triangle.

   c. Determine the previous row.

3. Determine the sum of the numbers in each horizontal row. What pattern do you see? Predict what the sum of the numbers in row 8 is. Verify this.

4. Each number in Pascal's Triangle can be written as a combination using the notation ${}_nC_r$, where n is the number of objects in the set and r is the number selected.

$n = 0$ (0th row)	${}_0C_0$
$n = 1$ (1st row)	${}_1C_0 \quad {}_1C_1$
$n = 2$ (2nd row)	${}_2C_0 \quad {}_2C_1 \quad {}_2C_2$
$n = 3$ (3rd row)	${}_3C_0 \quad {}_3C_1 \quad {}_3C_2 \quad {}_3C_3$
$n = 4$ (4th row)	${}_4C_0 \quad {}_4C_1 \quad {}_4C_2 \quad {}_4C_3 \quad {}_4C_4$
$n = 5$ (5th row)	${}_5C_0 \quad {}_5C_1 \quad {}_5C_2 \quad {}_5C_3 \quad {}_5C_4 \quad {}_5C_5$



Evaluate each of the combinations in the 4th row of the *combinations* triangle to confirm that these are the values found in the 4th row of Pascal's triangle.

5. Using combinations, determine

   a. the third element in the ninth row of Pascal's Triangle.

   b. the seventh element in the fifteenth row.

   c. the first five elements in row 20.

BINOMIAL EXPANSION

A binomial expression is the sum or difference of two terms. For example, (x + 1), (3x + 2y), and (a – b) are all binomial expressions.

You are already familiar with one process by which to expand a binomial, such as $(x+1)^2$ or $(x+1)^3$:

$$\begin{aligned}(x+1)^2 &= (x+1)(x+1)\\ &= x^2+x+x+1\\ &= x^2+2x+1\end{aligned}$$

$$\begin{aligned}(x+1)^3 &= (x+1)(x+1)^2\\ &= (x+1)(x^2+2x+1)\\ &= x^3+2x^2+x+x^2+2x+1\\ &= x^3+3x^2+3x+1\end{aligned}$$

However, if you want to determine the expansion of $(x+1)^6$, it is very cumbersome to do this by repeatedly multiplying (x + 1) by itself.

EXPLORING $(x+1)^n$

Pascal's triangle can be used to determine the coefficient of each term in the expansion of $(x+1)^n$. Study the table to the right.

Power	Binomial Expansion	Pascal's Triangle
2	$(x+1)^2 = \mathbf{1}x^2 + \mathbf{2}x + \mathbf{1}$	1, 2, 1
3	$(x+1)^3 = \mathbf{1}x^3 + \mathbf{3}x^2 + \mathbf{3}x + \mathbf{1}$	1, 3, 3, 1
4	$(x+1)^4 = \mathbf{1}x^4 + \mathbf{4}x^3 + \mathbf{6}x^2 + \mathbf{4}x + \mathbf{1}$	1, 4, 6, 4, 1
	... etc ...	

a. For each expansion, what happens to the exponent on x as you move through each term from left to right?

b. For each expansion, how are the coefficients related to Pascal's triangle?

c. Compare the number of terms in each expansion to the exponent to which the binomial is raised.

d. Expand $(x+1)^6$.

EXPLORING $(x+y)^n$

Study the following table:

$$(x+y)^0 = 1 \longleftarrow \text{1 term}$$
$$(x+y)^1 = x + y \longleftarrow \text{2 terms}$$
$$(x+y)^2 = x^2 + 2xy + y^2 \longleftarrow \text{3 terms}$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \longleftarrow \text{4 terms}$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \longleftarrow \text{5 terms}$$
$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 \leftarrow \text{6 terms}$$

a. What happens to the exponents on x as you move through each term from left to right?

b. What happens to the exponents on y as you move through each term from left to right?

c. Compare the number of terms in each expansion to the exponent to which the binomial is raised.

d. For the row $(x+y)^5$, determine the degree of each term in the expansion. Compare this to another row.

e. For each expansion, how are the coefficients related to Pascal's triangle?

Example 1: Expand Binomials

Use Pascal's triangle and the patterns observed above to expand $(a+b)^7$.

Solution:

THE BINOMIAL THEOREM

If we wanted to expand a binomial expression with a large power, for example $(x+y)^{32}$, the use of Pascal's triangle would not be recommended because of the need to generate a large number of rows of the triangle. An alternative method is to use the **binomial theorem**.

THE BINOMIAL THEOREM

$$(x+y)^n = {}_nC_0(x)^n(y)^0 + {}_nC_1(x)^{n-1}(y)^1 + {}_nC_2(x)^{n-2}(y)^2 + \ldots + {}_nC_{n-1}(x)^1(y)^{n-1} + {}_nC_n(x)^0(y)^n$$

Each term has the form ${}_nC_k(x)^{n-k}(y)^k$, where k+1 is the term number

Explore the expansion presented below:

$(2a+5)^4 =$

First term ${}_4C_0(2a)^4(5)^0 = 1(16a^4) = 16a^4$

Second term ${}_4C_1(2a)^3(5)^1 = 4(8a^3)(5) = 160a^3$

Third term ${}_4C_2(2a)^2(5)^2 = 6(4a^2)(25) = 600a^2$

Fourth term ${}_4C_3(2a)^1(5)^3 = 4(2a)(125) = 1000a$

Fifth term ${}_4C_4(2a)^0(5)^4 = 1(1)(625) = 625$

Full expansion: ____

Determining a specific term in a binomial expansion:

Example: Find the *eighth* term in the expansion of $(x+y)^{13}$.

Note that the *first* term of the expansion would be ${}_{13}C_0x^{13}y^0 = x^{13}$.

The *eighth* term in the expansion would be ______________ = ____________________

Example 2: Use the Binomial Theorem

a. Use the binomial theorem to expand $(3a-4b)^5$.

b. What is the third term in the expansion of $(5a^2+2)^6$?

c. In the expansion of $\left(a^2-\frac{2}{a}\right)^7$, which term, in simplified form, contains a^2?