The Factor and Rational Root Theorems

The Factor Theorem

The factor theorem states that (x–a) is a factor of a polynomial P(x) if and only if P(a)=0. In other words:

- If (x–a) is a factor of a polynomial P(x), then P(a)=0.
- If P(a)=0, then (x–a) is a factor of a polynomial P(x).

Extended version of the factor theorem: $(bx-a)$ is a factor of a polynomial P(x) if $P\left(\frac{a}{b}\right)=0$

For example,

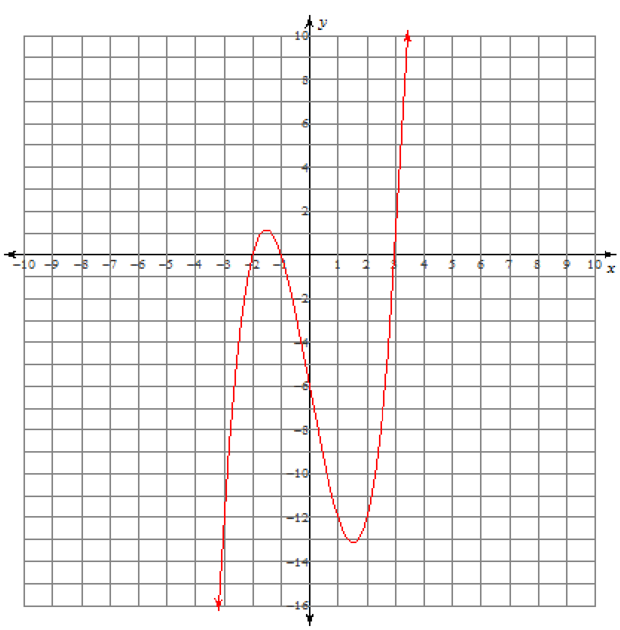
Given the polynomial, $P(x)=x^3-7x-6$, determine if (x–1) and (x+2) are factors by calculating P(1) and P(–2).

$$P(x)=x^3-7x-6$$
$$P(1)=1^3-7(1)-6$$
$$P(1)=1-7-6$$
$$P(1)=-12$$

Since P(1)=–12, then P(x) is not divisible by (x–1). Thus, (x–1) is not a factor of P(x).

$$P(x)=x^3-7x-6$$
$$P(-2)=(-2)^3-7(-2)-6$$
$$P(-2)=-8+14-6$$
$$P(-2)=0$$

Since P(–2)=0, then P(x) is divisible by (x+2). Thus, (x+2) is a factor of P(x).



The zeros of the polynomial function $P(x)=x^3-7x-6$ are related to the factors of the polynomial. The graph shows that the zeros of the function, or the x-intercepts of the graph, are at

x=–2, x=–1, and x= 3.

The corresponding factors of the polynomial are:
(x+2), (x+1), and (x–3).

Example 1: Use the Factor Theorem to Test for Factors of a Polynomial

Which binomials are factors of the polynomial $P(x) = x^3 + 4x^2 + x - 6$?

a. x – 1 b. x – 2 c. x + 2 d. x + 3

Solution:

$P(x) = x^3 + 4x^2 + x - 6$ Possible factor (x – 1)	$P(x) = x^3 + 4x^2 + x - 6$ Possible factor (x – 2)	$P(x) = x^3 + 4x^2 + x - 6$ Possible factor (x + 2)	$P(x) = x^3 + 4x^2 + x - 6$ Possible factor (x + 3)
(x–1) is / is not a factor of $P(x) = x^3 + 4x^2 + x - 6$	**(x–2) is / is not a factor of** $P(x) = x^3 + 4x^2 + x - 6$	**(x+2) is / is not a factor of** $P(x) = x^3 + 4x^2 + x - 6$	**(x+3) is / is not a factor of** $P(x) = x^3 + 4x^2 + x - 6$

Rational Zero (or Rational Root) Theorem

Given a polynomial function with integer coefficients, every rational zero will be of the form $\frac{p}{q}$, where *p* is a factor of the *constant term* and *q* is a factor of the *leading coefficient*.

For example, consider the polynomial function $f(x) = 6x^2 + 5x - 4$. The *easiest* method for obtaining the zeros of this function is to set $f(x) = 0$, factor, and solve for x:

$0 = 6x^2 + 5x - 4 \quad \rightarrow \quad 0 = (3x + 4)(2x - 1) \quad \rightarrow \quad x = -4/3$ or $x = 1/2$

However, to illustrate how to find the possible zeros using the *rational zero theorem*, the theorem states that, for this polynomial, any rational zero must have a factor of –4 (______________________) in the numerator and a factor of 6 (______________________________) in the denominator:

p: factors of –4 = ±1, ±2, ±4

q: factors of 6 = ±1, ±2, ±3, ±6

The *possibilities* of $\frac{p}{q}$, in simplest form, are: $\pm 1, \pm 1/2, \pm 1/3, \pm 1/6, \pm 2, \pm 2/3, \pm 4, \pm 4/3$

These values can be tested, by using direct substitution or by using synthetic or long division, to see if the remainder is equal to zero. If it is, then $\frac{p}{q}$ is a zero of the function. In this case, the zeros are $-4/3$ & $1/2$.

Of course, the rational zero theorem is *not* recommended for finding the zeros of a quadratic function, however, it is often used to find the zeros of higher-degree polynomial functions.

Example 2: Use the Rational Zero Theorem to Determine One of the Factors of a Polynomial with a Leading Coefficient of 1

For the polynomial function $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$: a. Find *all possible* rational roots of P(x).
b. Determine *one* of the factors of P(x).

Solution:

a. All *possible* rational roots of P(x) must have a factor of _____ (constant term) in the numerator and a factor of _____ (leading coefficient) in the denominator:

p: factors of 12 = ______, ______, ______, ______, ______, ______

q: factors of 1 = ______

The *possibilities* of $\frac{p}{q}$, in simplest form, are: ______, ______, ______, ______, ______, ______

b. Now determine one of the *actual* roots of $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$:

Since P(_____) = 0, then _______________ is a factor of P(x).

Example 3: Use the Rational Zero Theorem to Determine One of the Factors of a Polynomial with Leading Coefficient *not* equal to 1

Consider the polynomial function $P(x) = 2x^3 - 11x^2 + 12x + 9$: a. Find *all possible* rational roots of P(x).
b. Determine *one* of the factors of P(x).

Solution:

a. All *possible* rational roots of P(x) must have a factor of _____ (constant term) in the numerator and a factor of _____ (leading coefficient) in the denominator:

p: factors of 9 = ______, ______, ______

q: factors of 2 = ______, ______

The *possibilities* of $\frac{p}{q}$, in simplest form, are: ______, ______, ______, ______, ______, ______

b. Now determine one of the *actual* roots of $P(x) = 2x^3 - 11x^2 + 12x + 9$:

Since P(_____) = 0, then _______________ is a factor of P(x).

Example 4: Factor Using the Factor Theorem

Consider the polynomial expression $2x^3 - 3x^2 - 11x + 6$.

a. Completely factor the expression.
b. Use the factors to determine the zeros of the corresponding polynomial function.

Solution:

a. Let $P(x) = 2x^3 - 3x^2 - 11x + 6$

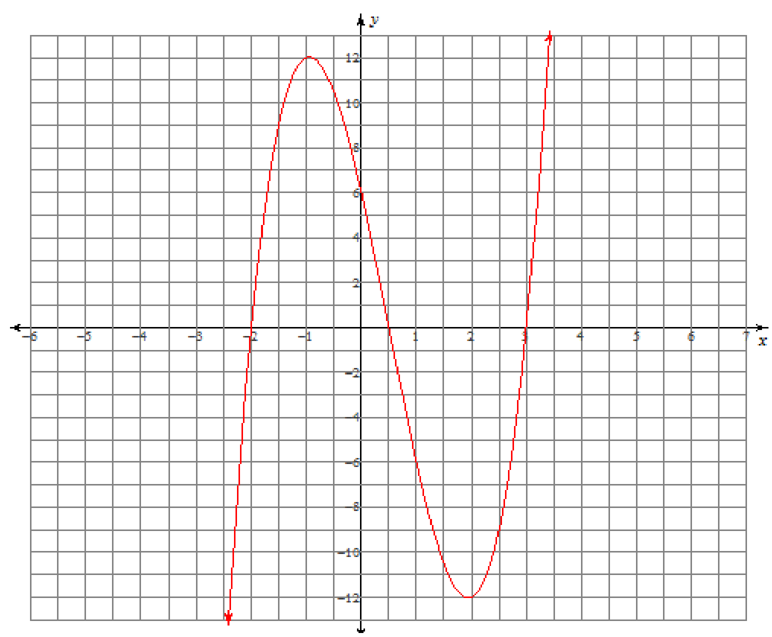
- Use the rational root and factor theorems to determine one of the roots of the polynomial.

- Since P(   )=0, then __________ is a factor of P(x).

- The remaining factor(s) can be determined by first using synthetic or long division and then following with other factoring methods.

- Factors of the polynomial expression $2x^3 - 3x^2 - 11x + 6$ are:
__________, __________, __________.

b. The zeros of $P(x) = 2x^3 - 3x^2 - 11x + 6$ are:
________, ________, and ________.



Example 5: Factor Higher-Degree Polynomials

Completely factor the polynomial $x^4 + 3x^3 - 7x^2 - 27x - 18$.

Solution:

Let $P(x) = x^4 + 3x^3 - 7x^2 - 27x - 18$

- Use the rational root and factor theorems to find one of the roots of the polynomial.

- Since P(   )=0, then __________ is a factor of P(x).

- The next factor can be determined by using synthetic or long division:

The remaining factor is ________________________________

- Let f(x)= ____

- Factor using one of the specialized factoring strategies, if possible. Or, use the rational root and factor theorems again with synthetic or long division to find the remaining factors of the polynomial.

- Combine all the factors to write the fully factored form:
$x^4 + 3x^3 - 7x^2 - 27x - 18 =$ ____

Example 6: Solve Problems Involving Polynomial Expressions

An artist creates a carving from a block of soapstone. The soapstone is in the shape of a rectangular prism whose volume, in cubic feet, is represented by $V(x) = 6x^3 + 25x^2 + 2x - 8$. What are the factors, in terms of x, that represent the possible dimensions of the block of soapstone?

Solution: