Exploring Rational Functions Using Transformations

- Rational functions are functions that can be written in the form $f(x)=\dfrac{p(x)}{q(x)}$, where p(x) and q(x) are polynomial expressions and $q(x)\neq 0$.
- Some examples of rational functions are $f(x)=\dfrac{5}{x}$, $f(x)=\dfrac{6}{x-3}+2$, and $f(x)=\dfrac{4}{x^2+10x+25}$.

Graphing a Rational Function

- You can graph a rational function by creating a *table of values* and then plotting these points. In the table of values, include the non-permissible value(s) as well as small and large values of x on either side of the non-permissible value(s) to give you a good spread of points on the graph.

- You can use what you know about the base functions $y=\dfrac{1}{x}$ and $y=\dfrac{1}{x^2}$ and apply *transformations* in order to graph functions of the form $y=\dfrac{a}{x-h}+k$ or $y=\dfrac{a}{(x-h)^2}+k$.

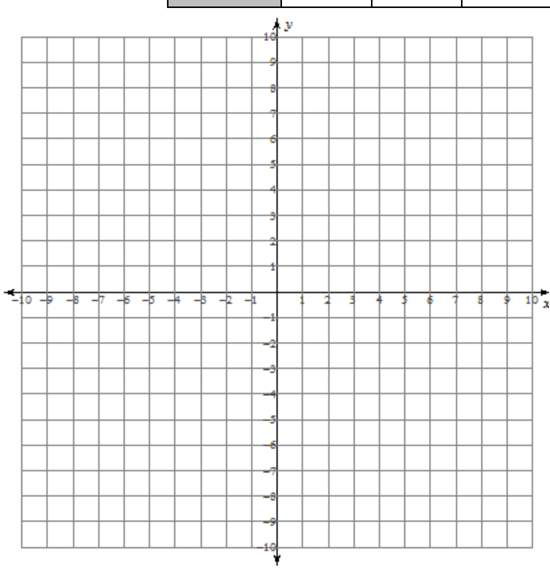
Example 1: Graph a Rational Function Using a Table of Values

Analyze the function $y=\dfrac{4}{x}$ using a table of values and a graph. Identify characteristics of the graph, including the behaviour of the function for its non-permissible value. Note that the graph of $y=\dfrac{4}{x}$ can be viewed as a

______________________________ of $y=\dfrac{1}{x}$ by a factor of _____.

Solution:

x	–10	–5	–2	–1	$\frac{-1}{2}$	$\frac{-1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	5	10
$y=\frac{4}{x}$							undefined						



Characteristic	$y=\frac{4}{x}$
Non-permissible value	
Equation of vertical asymptote	
Behaviour of the function as x approaches 0 *from the left*	$\lim_{x\to 0^-} f(x)=$
Behaviour of the function as x approaches 0 *from the right*	$\lim_{x\to 0^+} f(x)=$
End behaviour	$\lim_{x\to \pm\infty} f(x)=$
Equation of horizontal asymptote	
Domain	
Range	

Example 2: Graph a Rational Function Using Transformations

Sketch the graph of the function $y=\dfrac{3}{x+1}+5$ using transformations and identify any important characteristics of the graph.

Solution:

To obtain the graph of $y=\dfrac{3}{x+1}+5$ from the graph of $y=\dfrac{1}{x}$, apply the following transformations:

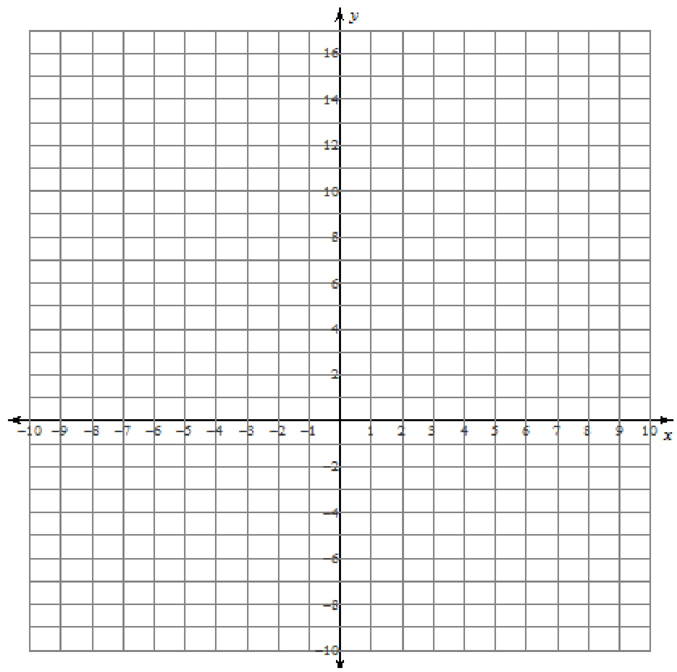
____

____

The vertical and horizontal asymptotes of the graph of $y=\dfrac{1}{x}$ are at x = 0 and y = 0. Therefore, for the transformed function, $y=\dfrac{3}{x+1}+5$, the vertical asymptote is located at __________ and the horizontal asymptote is located at __________.

Mapping Rule: $(x, y) \rightarrow ($ $\qquad$, $\qquad$ $)$

$y=\dfrac{1}{x}$			$y=\dfrac{3}{x+1}+5$	
–4	–0.25			
–2	–0.5			
–1	–1			
–0.5	–2			
0	undefined			
0.5	2			
1	1			
2	0.5			
4	0.25			



Characteristic	$y=\dfrac{3}{x+1}+5$
Non-permissible value	
Equation of vertical asymptote	
Behaviour of the function as x approaches –1 *from the left*	$\lim\limits_{x\to -1^-} f(x)=$
Behaviour of the function as x approaches –1 *from the right*	$\lim\limits_{x\to -1^+} f(x)=$
End behaviour	$\lim\limits_{x\to \pm\infty} f(x)=$
Equation of horizontal asymptote	
Domain	
Range	

Convert Functions from Rational Form to Transformational Form

Rational Form: $f(x)=\dfrac{n(x)}{d(x)}$ **Transformational Form:** $f(x)=\dfrac{a}{x-h}+k$

Example:

Write $f(x)=\dfrac{7x+6}{x+1}$ in the form $f(x)=\dfrac{a}{x-h}+k$.

$$\begin{array}{r} 7 \leftarrow \text{Quotient} \\ x+1\overline{)7x+6} \\ -(7x+7) \\ \hline -1 \xleftarrow{\text{Remainder}} \end{array}$$

$$f(x)=\underbrace{7}_{\text{Quotient}}+\frac{-1}{x+1}$$

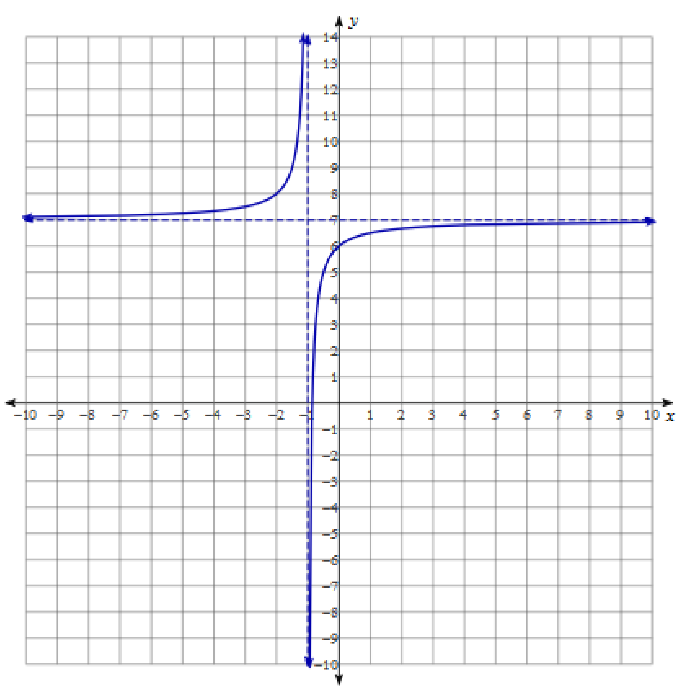
Divide using long division. The answer is the quotient, plus the remainder over the original divisor (or, the denominator).

Note:

The answer can also be written as $f(x)=\dfrac{-1}{x+1}+7$

$f(x)=\dfrac{7x+6}{x+1}$

$f(x)=\dfrac{-1}{x+1}+7$



Example 3: Graph a Rational Function with Linear Expressions in the Numerator and the Denominator

Graph the function $f(x) = \dfrac{3x+4}{x-2}$. Identify any asymptotes and intercepts.

Solution:

Determine the location of the *intercepts* and *asymptotes* first, and use them as a guide to sketch the graph.

y-intercept of the function $y = \dfrac{3x+4}{x-2}$:

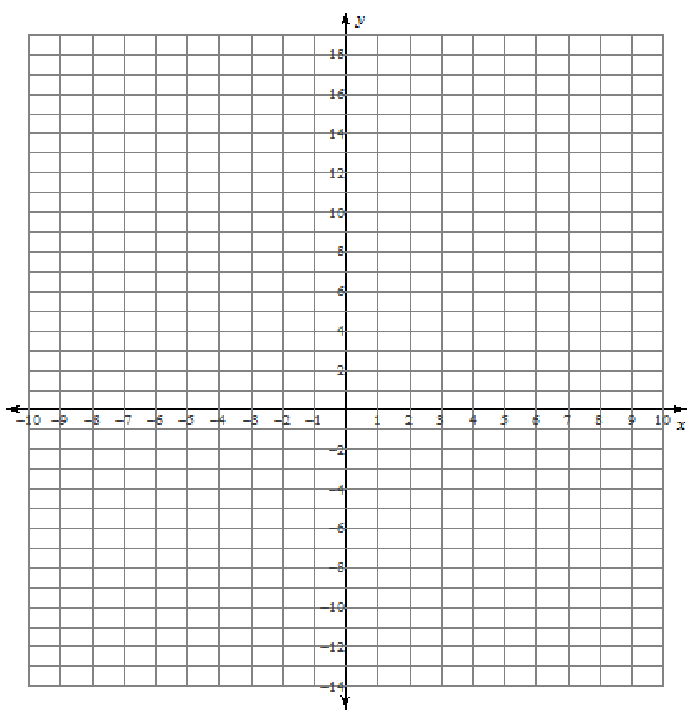
x-intercept of the function $y = \dfrac{3x+4}{x-2}$:

y-intercept: __________

x-intercept: __________

Using *long division*, we can write the function $y = \dfrac{3x+4}{x-2}$ in the form $y = \dfrac{a}{x-h} + k$, which reveals the location of both the vertical and horizontal asymptotes.

$$y = \frac{3x+4}{x-2}$$



Required transformations of $y = \dfrac{1}{x}$:

____

____

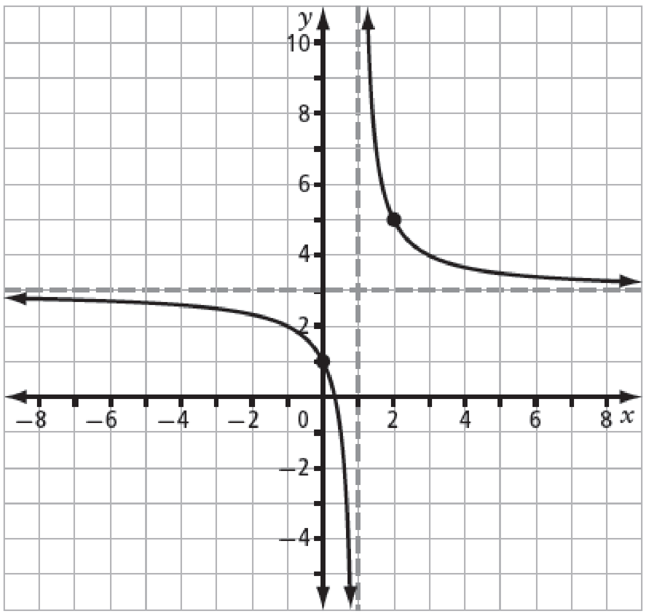
Vertical asymptote: __________

Horizontal asymptote: __________

$(1,\ 1) \to$ __________ $(-1,\ -1) \to$ __________

Example 4: Determining the Equation of a Rational Function

Determine the equation, in the form $y = \dfrac{a}{x-h} + k$, of the following rational function.



Solution:

From the location of the vertical and horizontal asymptotes, we can determine that the graph of $y = \dfrac{1}{x}$ has been translated ______________________________ and ______________________________.

We can, therefore, substitute the *h* and *k* values into the transformational form of the equation:

$$y = \frac{a}{x-h} + k \ =$$

Then, use one of the given points to find the value of *a*:

$$y = \frac{a}{x-1} + 3$$

Equation of the function: ________________________

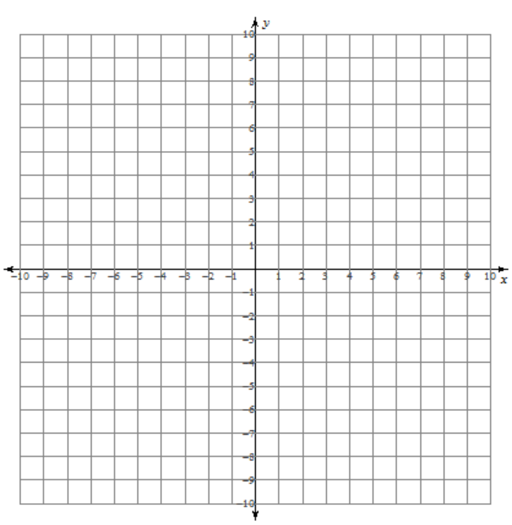
Example 5: Compare Rational Functions

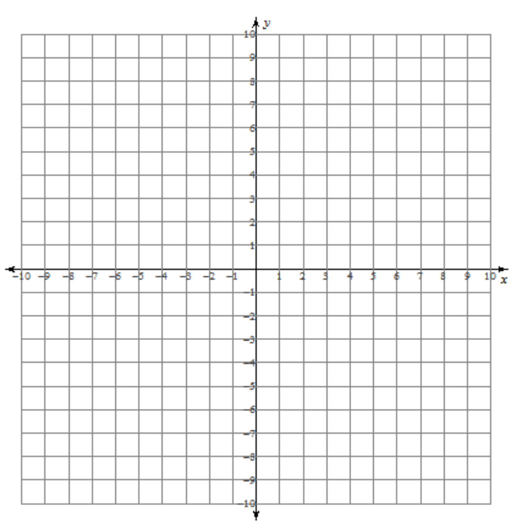
Graph the base function (a) and use transformations to graph functions (b) and (c).

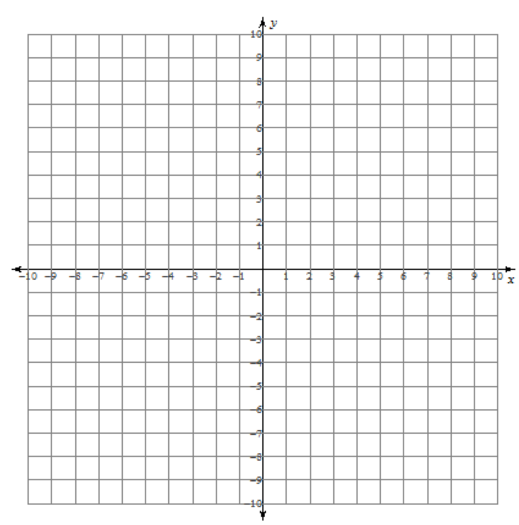
a. $y=\dfrac{1}{x^2}$ b. $y=\dfrac{3}{x^2-8x+16}$ c. $y=5-\dfrac{1}{(x+2)^2}$

Solution:

Characteristic	$y=\dfrac{1}{x^2}$	$y=\dfrac{3}{x^2-8x+16}$ Factored Form:	$y=5-\dfrac{1}{(x+2)^2}$
Non-permissible value			
Equation of vertical asymptote			
Behaviour of the function as x approaches the non-permissible value *from the left*	$\lim\limits_{x\to 0^-} f(x)=$	$\lim\limits_{x\to 4^-} f(x)=$	$\lim\limits_{x\to -2^-} f(x)=$
Behaviour of the function as x approaches the non-permissible value *from the right*	$\lim\limits_{x\to 0^+} f(x)=$	$\lim\limits_{x\to 4^+} f(x)=$	$\lim\limits_{x\to -2^+} f(x)=$
End Behaviour	$\lim\limits_{x\to \pm\infty} f(x)=$	$\lim\limits_{x\to \pm\infty} f(x)=$	$\lim\limits_{x\to \pm\infty} f(x)=$
Equation of horizontal asymptote			
Domain			
Range			







Example 6: Apply Rational Functions

Victoria is producing a tourism booklet for the city of Fredericton and its surrounding area. She is comparing the cost of printing from two different companies. The first company (A) charges a \$50 setup fee and \$2.50 per booklet. The second company (B) charges \$80 for the setup and \$2.10 per booklet.

a. Write the average cost per booklet for each company as a function of the number of booklets printed.
b. Graph the two functions.
c. Explain how the characteristics of the graphs are related to the situation.
d. Give Victoria advice about how she should choose a printing company.

Solution:

a. Let x = ____

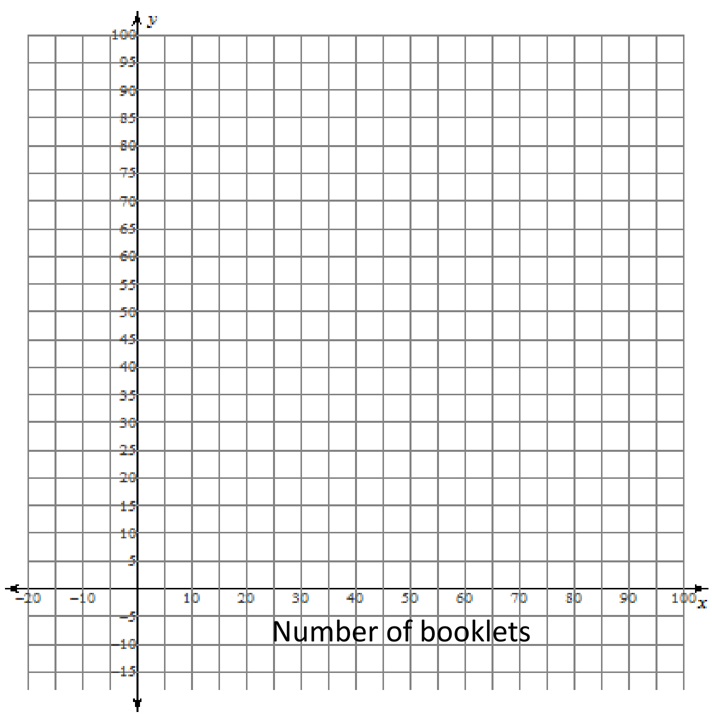
Company A: **Company B:**

C(x) = **C(x) =**

= **=**

b. Use the graphing calculator to graph the two functions.

Cost /booklet



c. The graph shows that the price per booklet is much _________________________ if you buy in small quantities, but gets progressively ___________________________ the _______________ you buy.

d. For more than ________ booklets, Victoria should buy from company ______. For fewer, she should buy from company ______.