Understanding Limits

In algebra, we are often required to calculate an exact value of a function y=f(x). When determining a specific y-value of a function, we simply need the corresponding x-value.

However, what if the function isn't just a simple curve? For example, we will be examining functions with holes, functions that shoot up toward infinity, functions that oscillate, etc. To describe the behaviour of the y-values of such functions, we can use a *limit*.

Intuitive Definition of a Limit

The limit of a function describes how a function behaves ***near*** a specific point, not ***at*** that point. If the values of y=f(x) get closer and closer to one number, L, as we take values of x very close to (but not equal to) a number, c, then we *say* **"The limit of f(x), as x approaches c, is L"** and we *write*:

$$\lim_{x\to c} f(x) = L$$

It is important to note that the value of the *limit* at x = c does *not* depend on the value of the *function* at x = c.

- f(c) is a single number that describes the value of f(x) ***at*** point x = c.
- $\lim_{x\to c} f(x)$ is a single number that describes the behavior of f(x) ***near, but not at***, the point x = c.

Using Tables to Find Limits

One way to approximate a limit is to use a table of values. Choose x-values close to c from both the left and the right. If the y-values approach a distinct value, L, as x approaches c, then the limit is L.

Example 1:

Determine the value of $\lim_{x\to 5} \dfrac{x^2-3x-10}{x-5}$

Solution:

1. Create a table of values for the function $f(x) = \dfrac{x^2-3x-10}{x-5}$ and fill in values of x that get close to 5 from both sides. Then, calculate the corresponding f(x) values.

x	4	4.5	4.9	4.95	5	5.05	5.1	5.5	6
f(x)									

2. Examine the table to determine if the f(x) values approach the same number from the left and the right.

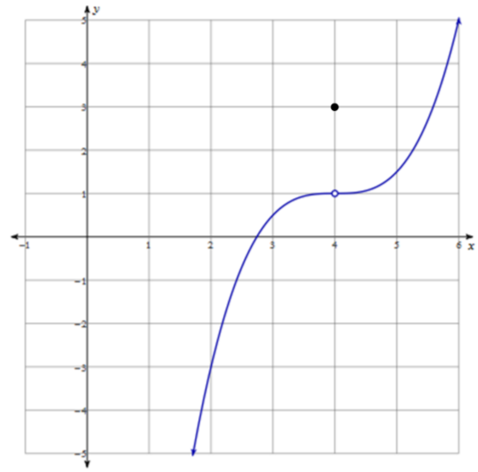
$$\lim_{x\to 5} \frac{x^2-3x-10}{x-5} = \_\_\_\_\_\_$$

* Note that we cannot find the limit by substituting x = 5 into the function since this results in division by zero.

Using Graphs to Find Limits

If we have the graph of a function f(x) near x = c, then it is usually easy to determine $\lim_{x \to c} f(x)$.

Example 2:
Use the graph of the function to the right to determine the following limits.



a. $\lim_{x \to 4} f(x)$ b. $\lim_{x \to 2} f(x)$

Solution:

a. As the x-values get closer to 4 from the left and from the right (We will learn more about Left- & Right-Hand Limits later), the y-values get closer to _____.

Note that f(4) = _____, but the limit of f(x) as x approaches 4 is _____.

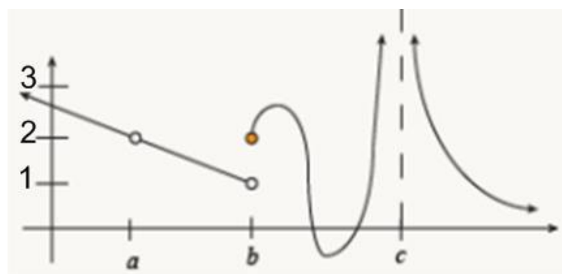
$\lim_{x \to 4} f(x) =$ _____

b. As the x-values get closer to 2 from the left and from the right, the y-values get closer to _____.

Note that f(2) = _____. In this case, the function value and the limit value are the same.

$\lim_{x \to 2} f(x) =$ _____

Example 3:
Use the graph of the function to the right to determine the following limits.



a. $\lim_{x \to a} f(x)$ b. $\lim_{x \to b} f(x)$ c. $\lim_{x \to c} f(x)$

Solution:

a. As the x-values get closer to *a* from either side, the y-values get closer to _____.

Note that the function doesn't exist at *a*, but the limit does.

$\lim_{x \to a} f(x) =$ _____

b. As the x-values get closer to *b* from the left, the y-values get closer to _____.
As the x-values get closer to *b* from the right, the y-values get closer to _____.

Since the function approaches different values from each side, the limit *does not exist*.

$\lim_{x \to b} f(x)$ __________

c. As the x-values get closer to *c* from either side, the y-values get closer to ________.

Since the function approaches infinity from each side, the limit *does not exist*.

$\lim_{x \to c} f(x)$ __________

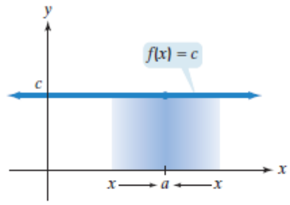
Calculating Limits Algebraically

Limit Rules

If n is a positive integer, *c* is a constant, and f(x) and g(x) are functions that have limits at x = a, then the following rules hold:

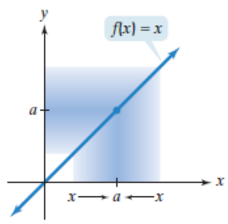
1. **Constant Rule**

$$\lim_{x\to a} c = c$$



2. **Identity Rule**

$$\lim_{x\to a} x = a$$



3. **Constant Multiple Rule**

$$\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$$

4. **Sum Rule**

$$\lim_{x\to a}\left[f(x)+g(x)\right] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

5. **Difference Rule**

$$\lim_{x\to a}\left[f(x)-g(x)\right] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

6. **Product Rule**

$$\lim_{x\to a}\left[f(x)\cdot g(x)\right] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$$

7. **Quotient Rule**

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)},\ \text{if } \lim_{x\to a} g(x)\neq 0$$

8. **Power Rule**

$$\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$$

9. **n^{th} Root Rule**

$$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)},\ \text{if } \lim_{x\to a} f(x) > 0$$

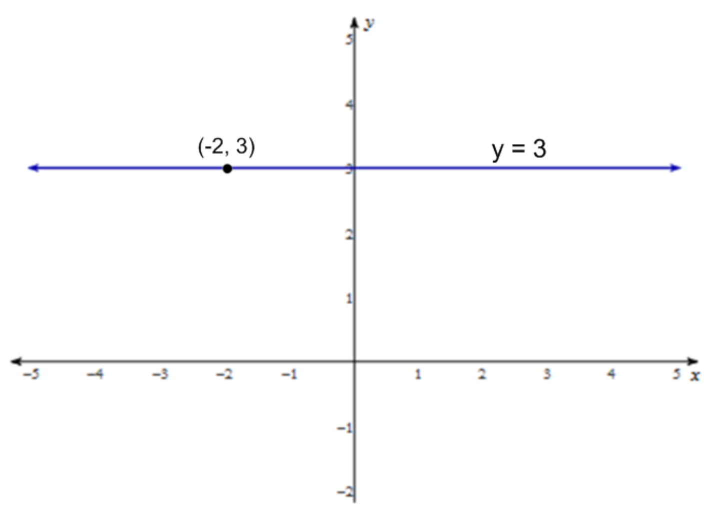
Example 4: Finding a Limit Using Limit Rules

Find $\lim_{x\to -2} 3$

Solution:

Use the Constant Rule for limits.

$\lim_{x\to -2} 3 =$ _____



Look at the graph of the function y = 3 to see why the limit of the constant function as x approaches 3 is the same as the constant value. Notice that for this function, the limit as x approaches *any* value is 3.

Example 5: Finding a Limit Using Limit Rules

Find $\lim_{x\to 3}\left(2x^2 - 4x - 1\right)$

Solution:

Use the Difference Rule:

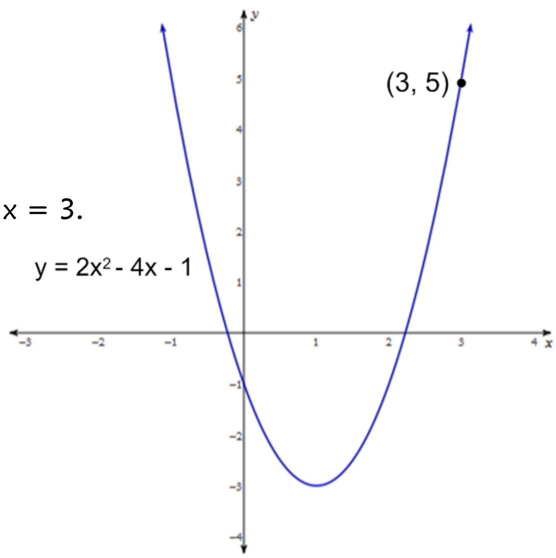
Use the Constant Multiple Rule:

Use the Power Rule:

Use the Constant Rule:

Use the Identity Rule:

Simplify:



Note that the limit as x approaches 3 is the same as the function value at x = 3.
This is always the case for limits of polynomial functions.
We can use *direct substitution* to evaluate all polynomial limits.

Example 6: Finding a Limit by Direct Substitution

a. Find $\lim_{x\to -1}\left(-x^4 + 5x^3 + 3x^2 - 1\right)$ b. Find $\lim_{x\to 8}\dfrac{x^2 - 2x - 3}{x+2}$

Solution:

a. Use direct substitution.

$$\lim_{x\to -1}\left(-x^4 + 5x^3 + 3x^2 - 1\right) =$$

b. Use direct substitution.

$$\lim_{x\to 8}\frac{x^2 - 2x - 3}{x+2} =$$

Example 7: Finding a Limit by Factoring & Simplifying First

a. Find $\lim\limits_{x\to1}\dfrac{2x^2-x-1}{x-1}$ b. Find $\lim\limits_{x\to2}\dfrac{x^3-8}{x-2}$

Solution:

a. If we try to evaluate $\lim\limits_{x\to1}\dfrac{2x^2-x-1}{x-1}$ by using direct substitution, we get $\dfrac{2(1)^2-1-1}{1-1}=\dfrac{0}{0}$.

So, factor and simplify first, *then* use substitution.

$$\lim_{x\to1}\frac{2x^2-x-1}{x-1}=$$

b. If we try to evaluate $\lim\limits_{x\to2}\dfrac{x^3-8}{x-2}$ by using direct substitution, we get $\dfrac{(2)^3-8}{2-2}=\dfrac{0}{0}$.

So, factor and simplify first, *then* use substitution.

$$\lim_{x\to2}\frac{x^3-8}{x-2}=$$

Example 8: Finding a Limit by Using the Conjugate Technique

a. Find $\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2}$ b. Find $\lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3}$ c. Find $\lim_{x\to 0} \frac{\sqrt[3]{x+27}-3}{x}$

Solution:

a. Direct substitution yields $\frac{0}{0}$. So, multiply by the conjugate form of 1.

$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} =$$

b. Direct substitution yields $\frac{0}{0}$. So, multiply by the conjugate form of 1.

$$\lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3} =$$

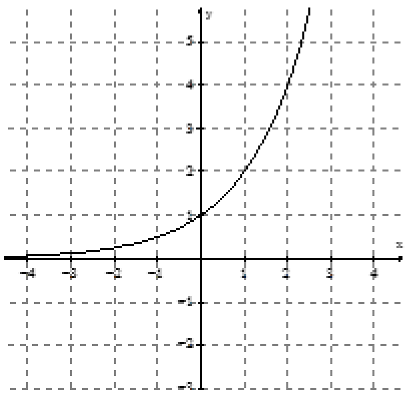
c. Direct substitution yields $\frac{0}{0}$. So, multiply by the conjugate form of 1.

$$\lim_{x \to 0} \frac{\sqrt[3]{x+27}-3}{x} =$$

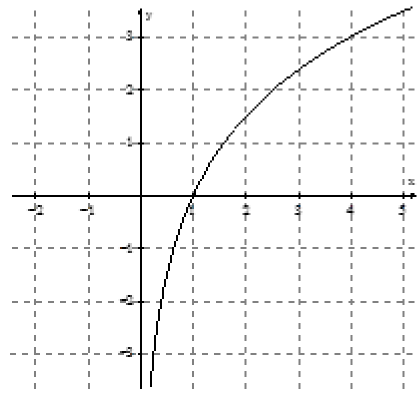
Practice:

1. Determine the following limits graphically.

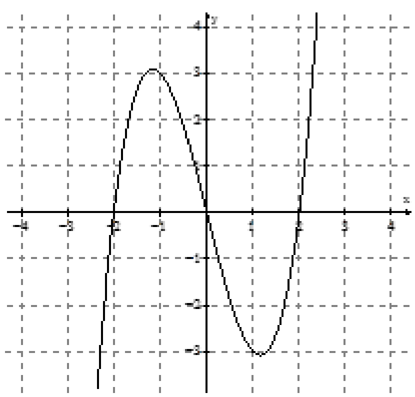
a. $\lim_{x \to 1} f(x)$



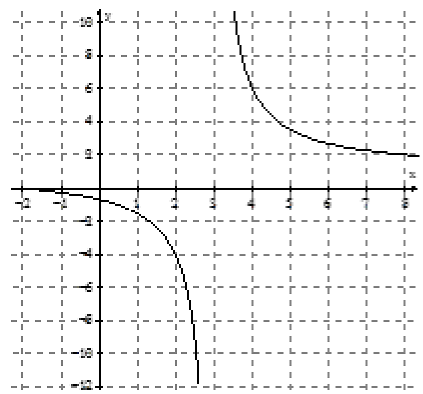
b. $\lim_{x \to 1} f(x)$



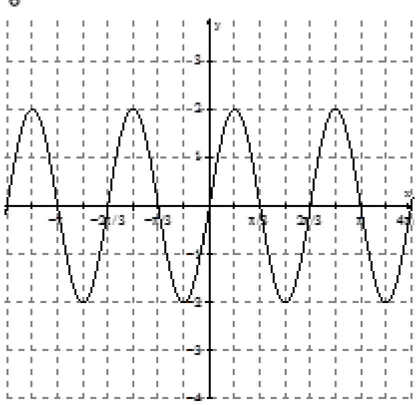
c. $\lim_{x \to -1} f(x)$



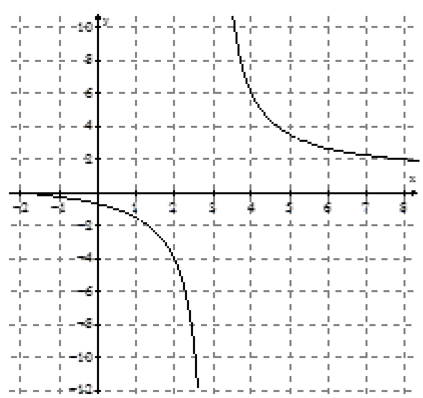
d. $\lim_{x \to 3} f(x)$



e. $\lim_{x \to \frac{\pi}{6}} f(x)$



f. $\lim_{x \to 4} f(x)$



2. Use the *Limit Rules* to calculate $\lim_{x \to 3} \dfrac{(4x+2)^2}{\sqrt[3]{1-x^2}}$.

3. Evaluate the following limits.

a) $\lim\limits_{x\to 4} \dfrac{x-4}{x^3-64}$

b) $\lim\limits_{x\to 7} \dfrac{\sqrt[5]{3-5x}}{(x-5)^3}$

c) $\lim\limits_{x\to -3} \dfrac{x+3}{3-\sqrt{x+12}}$

d) $\lim\limits_{x\to -3} \sqrt[3]{\dfrac{x-4}{6x^2+2}}$

e) $\lim\limits_{x\to 3} \dfrac{\frac{1}{x^2}-\frac{1}{9}}{x-3}$

f) $\lim\limits_{h\to 0} \dfrac{(-3+h)^2-9}{h}$

g) $\lim\limits_{h\to 0} \dfrac{(2+h)^3-8}{h}$

h) $\lim\limits_{x\to 0} \dfrac{\frac{1}{1+x}-1}{x}$

i) $\lim\limits_{x\to -1} \dfrac{2x+3}{3x+2}$

j) $\lim\limits_{x\to -8} \dfrac{x+8}{\sqrt[3]{x}+2}$

k) $\lim\limits_{x\to -1} \dfrac{8}{(3+x)\sqrt{3-x}}$

l) $\lim\limits_{x\to 9} \dfrac{x-9}{3-\sqrt{x}}$

m) $\lim\limits_{x\to 0} \dfrac{(3+x)^3-3^3}{(3+x)^2-3^2}$

n) $\lim\limits_{x\to 9} \dfrac{x^2-81}{3-\sqrt{x}}$

o) $\lim\limits_{x\to 16} \dfrac{2\sqrt{x}+x^{3/2}}{\sqrt[4]{x}+5}$

p) $\lim\limits_{x\to 2} \dfrac{x^2-4}{3x^2-7x+2}$

q) $\lim\limits_{x\to 7} \dfrac{\sqrt{x+2}-3}{x-7}$

r) $\lim\limits_{x\to 4} \left(\frac{8}{x^2-16}-\frac{1}{x-4}\right)$

s) $\lim\limits_{x\to 5} \dfrac{2x}{x-5}$

t) $\lim\limits_{x\to 0} \dfrac{x^3}{2x^3+3x^4}$

u) $\lim\limits_{x\to 2} \dfrac{x^2-2x}{2x^2-7x+6}$

v) $\lim\limits_{x\to \pi} \dfrac{1+\sin x}{\cos x}$

w) $\lim\limits_{x\to \frac{\pi}{4}} ((\cos x)+(\sin x))$

x) $\lim\limits_{x\to \frac{3\pi}{2}} \dfrac{\sin x}{\cos x}$

y) $\lim\limits_{x\to 4} \dfrac{2x^3-128}{\sqrt{x}-2}$

z) $\lim\limits_{h\to 0} \dfrac{\sqrt[3]{h+1}-1}{h}$

Answers:

1. a. 2 b. 0 c. 3 d. DNE e. 2 f. 6

2.

$$\lim_{x\to 3}\frac{(4x+2)^2}{\sqrt[3]{1-x^2}} = \frac{\lim\limits_{x\to 3}(4x+2)^2}{\lim\limits_{x\to 3}\sqrt[3]{1-x^2}} = \frac{\left(\lim\limits_{x\to 3}(4x+2)\right)^2}{\sqrt[3]{\lim\limits_{x\to 3}(1-x^2)}} = \frac{\left(\lim\limits_{x\to 3}(4x)+\lim\limits_{x\to 3}2\right)^2}{\sqrt[3]{\lim\limits_{x\to 3}1-\lim\limits_{x\to 3}(x^2)}} = \frac{\left(4\lim\limits_{x\to 3}x+\lim\limits_{x\to 3}2\right)^2}{\sqrt[3]{\lim\limits_{x\to 3}1-\left(\lim\limits_{x\to 3}x\right)^2}}$$

$$= \frac{(4(3)+2)^2}{\sqrt[3]{1-(3)^2}}$$

$$= \frac{196}{\sqrt[3]{-8}} = \frac{196}{-2} = -98$$

3. **a)** $\frac{1}{48}$ **b)** $-\frac{1}{4}$ **c)** -6 **d)** $-\frac{1}{2}$ **e)** $-\frac{2}{27}$ **f)** -6 **g)** 12 **h)** -1

i) -1 **j)** 12 **k)** 2 **l)** -6 **m)** $\frac{9}{2}$ **n)** -108 **o)** $\frac{72}{7}$ **p)** $\frac{4}{5}$ **q)** $\frac{1}{6}$

r) $-\frac{1}{8}$ **s)** DNE **t)** $\frac{1}{2}$ **u)** 2 **v)** -1 **w)** $\sqrt{2}$ **x)** DNE **y)** 384 **z)** $\frac{1}{3}$