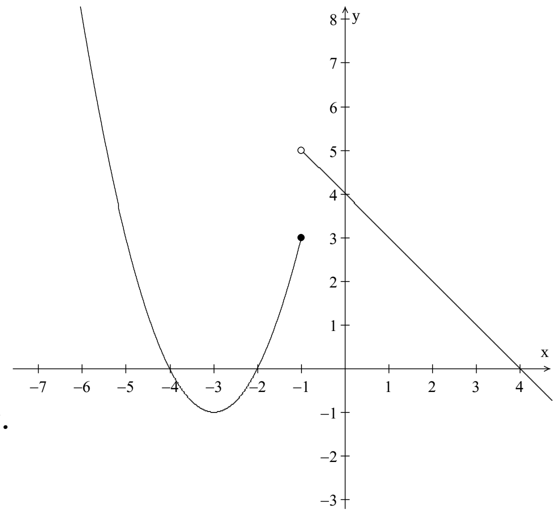
Left-Hand and Right-Hand Limits

The graph of following the piecewise-defined function shows that there is a break in the graph.

$$f(x)=\begin{cases} x^2+6x+8 & x\le -1 \\ -x+4 & x>-1 \end{cases}$$



Notice that as x approaches –1 *from the left*, the f(x) values are approaching _____, but as x approaches –1 *from the right*, the f(x) values are approaching _____. Since the f(x) values do not approach the *same* number as x approaches –1 from the left and from the right, the limit *does not exist* at x= –1.

However, even though the limit does not exist at x = –1, we can still describe the behaviour of the graph using one-sided limits. These one-sided limits have their own special notation.

Left-Hand Limit Notation: $\lim\limits_{x\to c^-} f(x)$ "the limit of f(x) as x approaches c from the left"

Right-Hand Limit Notation: $\lim\limits_{x\to c^+} f(x)$ "the limit of f(x) as x approaches c from the right"

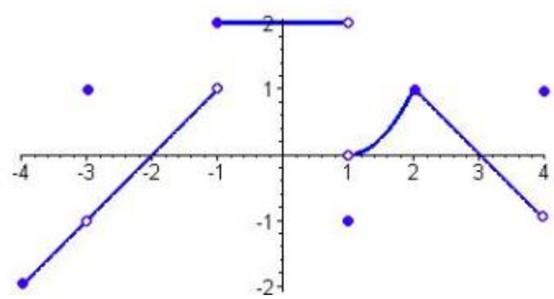
One-Sided Limit Theorem

$\lim\limits_{x\to c} f(x)=L$ if and only if $\lim\limits_{x\to c^-} f(x)=\lim\limits_{x\to c^+} f(x)=L$

We can use this theorem in two different ways:

1. If $\lim\limits_{x\to c^-} f(x)\ne\lim\limits_{x\to c^+} f(x)$, then $\lim\limits_{x\to c} f(x)$ does not exist.
2. If $\lim\limits_{x\to c^-} f(x)=\lim\limits_{x\to c^+} f(x)=L$, then $\lim\limits_{x\to c} f(x)=L$

Example 1: Finding One-Sided Limits from a Graph



Find the following limits, if they exist.

a. $\lim\limits_{x\to -3} f(x)$ b. $\lim\limits_{x\to 1} f(x)$

Solution:

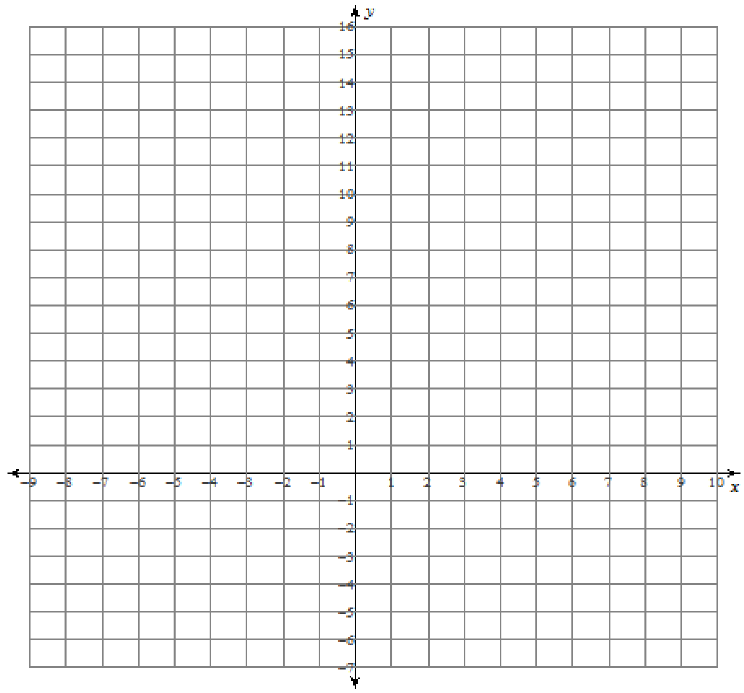
Use the graph of f(x) shown above. For the limit to exist, the left-and right-hand limits must equal each other.

a. $\lim\limits_{x\to -3^-} f(x)=$ _____ $\lim\limits_{x\to -3^+} f(x)=$ _____ Since $\lim\limits_{x\to -3^-} f(x)=\lim\limits_{x\to -3^+} f(x)$, then $\lim\limits_{x\to -3} f(x)=$ _____

b. $\lim\limits_{x\to 1^-} f(x)=$ _____ $\lim\limits_{x\to 1^+} f(x)=$ _____ Since $\lim\limits_{x\to 1^-} f(x)\ne\lim\limits_{x\to 1^+} f(x)$, then $\lim\limits_{x\to 1} f(x)=$ ________

Example 2: Sketching Graphs of Piecewise Functions and Determining One-Sided Limits

a. Given the piecewise function: $f(x)=\begin{cases} x^2+5, \text{ if } x<2 \\ 3x+1, \text{ if } x \geq 2 \end{cases}$

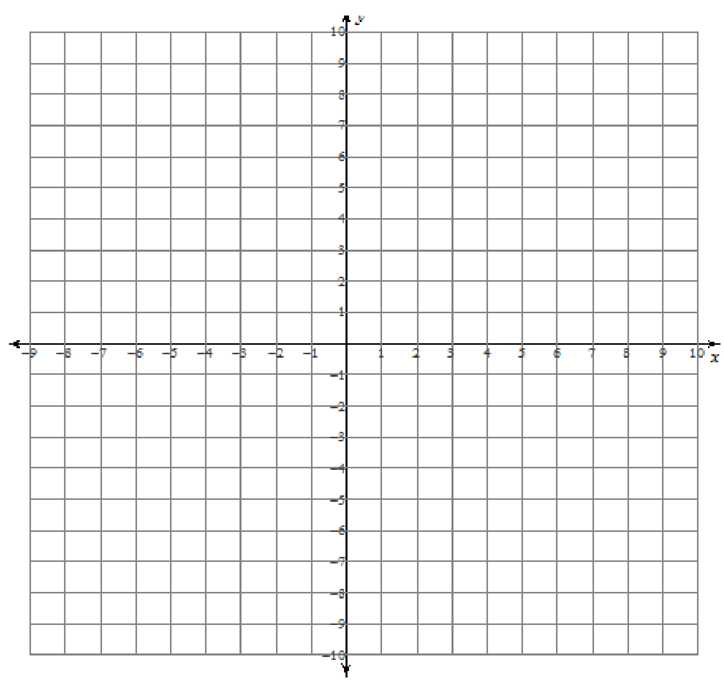


i. Sketch the graph of $f(x)$.

ii. Find the following limits, if they exist.

- $\lim\limits_{x \to 2^-} f(x)$ = _______
- $\lim\limits_{x \to 2^+} f(x)$ = _______
- $\lim\limits_{x \to 2} f(x)$ _________________

b. Given the piecewise function: $f(x)=\begin{cases} -x^2-4x-2, & x \leq 0 \\ (x-1)^3-1, & x>0 \end{cases}$



i. Sketch the graph of f(x).

ii. Find the following limits, if they exist.

- $\lim\limits_{x \to 0^-} f(x)$ = _______
- $\lim\limits_{x \to 0^+} f(x)$ = _______
- $\lim\limits_{x \to 0} f(x)$ = _______

Example 3: Finding One-Sided Limits Given the Equation of a Function

Given the piecewise function $f(x)=\begin{cases} 2x-1\ , & x \le -1 \\ x^2+1\ , & -1 < x \le 1 \\ -x+3\ , & x > 1 \end{cases}$, determine the following limits, if they exist.

a. $\lim\limits_{x \to -1} f(x)$ b. $\lim\limits_{x \to 1} f(x)$

Solution:

a. Step 1: Calculate the left-hand limit

$\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^-} (\_\_\_\_\_\_\_\_) =$

Step 2: Calculate the right-hand limit

$\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^+} (\_\_\_\_\_\_\_\_) =$

Step 3: Compare the left- and right-hand limits

Since $\lim\limits_{x \to -1^-} f(x) \ne \lim\limits_{x \to -1^+} f(x)$, then $\lim\limits_{x \to -1} f(x) =$ ________

b. Step 1: Calculate the left-hand limit

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (\_\_\_\_\_\_\_\_) =$

Step 2: Calculate the right-hand limit

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (\_\_\_\_\_\_\_\_) =$

Step 3: Compare the left- and right-hand limits

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x)$, so $\lim\limits_{x \to -1} f(x) =$ ______

Example 4: Finding One-Sided Limits Given the Equation of a Function

Rewrite $f(x)=\dfrac{2|x-6|}{x-6}$ as a piecewise function, and then determine $\lim\limits_{x \to 6} f(x)$, if it exists.

Solution:

Recall that $|x-6| =$

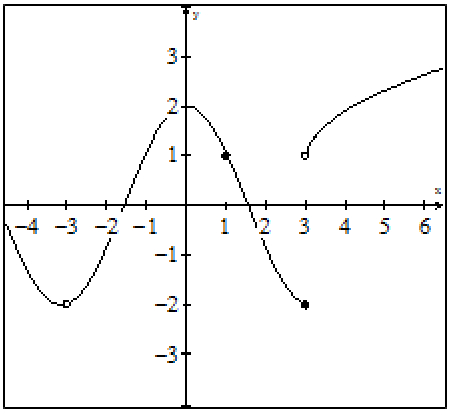
So, $f(x)=\dfrac{2|x-6|}{x-6} =$

$\lim\limits_{x \to 6^-} f(x) =$ ______ $\lim\limits_{x \to 6^+} f(x) =$ ______

Since $\lim\limits_{x \to 6^-} f(x) \ne \lim\limits_{x \to 6^+} f(x)$, then $\lim\limits_{x \to 6} f(x) =$ ____________

Practice:

1. Use the graph shown to determine the following limits.



a) $\lim\limits_{x\to -3^-} f(x)$ b) $\lim\limits_{x\to -3^+} f(x)$ c) $\lim\limits_{x\to -3} f(x)$

d) $\lim\limits_{x\to 1^+} f(x)$ e) $\lim\limits_{x\to 1^-} f(x)$ f) $\lim\limits_{x\to 1} f(x)$

g) $\lim\limits_{x\to 3^-} f(x)$ h) $\lim\limits_{x\to 3^+} f(x)$ i) $\lim\limits_{x\to 3} f(x)$

j) $f(-3)$ k) $f(1)$ l) $f(3)$

2. Sketch each piecewise function. Determine the given limit, if it exists. If the limit does not exist, provide an explanation.

a. $f(x)=\begin{cases} 2 & , \quad x<1 \\ 3 & , \quad x=1 \\ x+1, & \quad x>1 \end{cases}$ Find $\lim\limits_{x\to 1} f(x)$

b. $f(x)=\begin{cases} 4-x^2 , & -2<x\le 2 \\ x-2 , & x>2 \end{cases}$ Find $\lim\limits_{x\to 2} f(x)$

c. $f(x)=\begin{cases} |x+2|+1 , & x<-1 \\ -x+1 & ,-1\le x\le 1 \\ x^2-2x+2 , & x>1 \end{cases}$ Find $\lim\limits_{x\to 1} f(x)$

3. Determine the given limit, if it exists. If the limit does not exist, provide an explanation.

a) $f(x)=\begin{Bmatrix} 2x-1, & x\le -2 \\ -x+2, & x>-2 \end{Bmatrix}$ *Find* $\lim\limits_{x\to -2^-} f(x)$

b) $f(x)=\begin{Bmatrix} -x^2+4x-3, & x<1 \\ x-7, & x\ge 1 \end{Bmatrix}$ *Find* $\lim\limits_{x\to 1^-} f(x)$

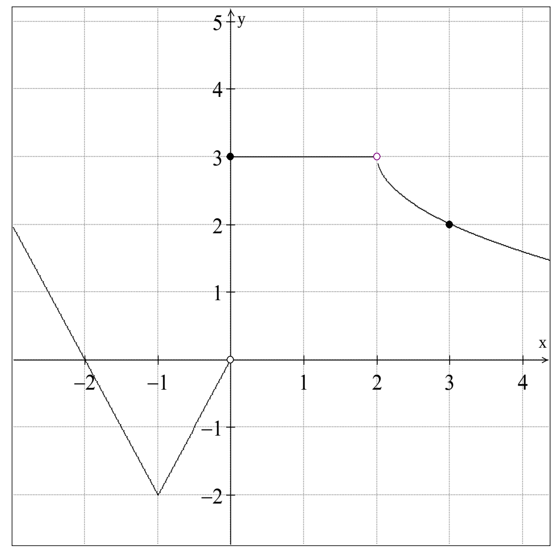
c) $f(x)=\begin{Bmatrix} x^2-2x+1, & x<-1 \\ -\dfrac{x}{2}+\dfrac{7}{2}, & x\ge -1 \end{Bmatrix}$ *Find* $\lim\limits_{x\to -1} f(x)$

d) $f(x)=\begin{Bmatrix} x+3, & x\in(-\infty,0] \\ -x+2, & x\in(0,2) \\ (x-2)^2, & x\in[2,\infty) \end{Bmatrix}$ *Find* $\lim\limits_{x\to 0} f(x)$ *and* $\lim\limits_{x\to 2} f(x)$

e) $f(x)=\begin{Bmatrix} (x+1)^2-1, & -2\le x<0 \\ \dfrac{5}{4}\sin\left(\dfrac{\pi x}{2}\right), & 0\le x<2 \\ (x-3)^2-1, & 2\le x\le 4 \end{Bmatrix}$ *Find* $\lim\limits_{x\to 2} f(x)$

4. Rewrite $f(x)=\dfrac{3|-x+2|}{-x+2}$ as a piecewise function and then determine the following limits.

a) $\lim\limits_{x\to 2^-} f(x)$ b) $\lim\limits_{x\to 2^+} f(x)$ c) $\lim\limits_{x\to 2} f(x)$



5. Consider the graph of a piecewise function shown to the right.

a. What is a possible equation for this function?

b. i) Determine $\lim\limits_{x\to 0} f(x)$, if it exists.

ii) Determine $\lim\limits_{x\to 2} f(x)$, if it exists.

6. The function f(t) is defined by $f(t)=\begin{cases}3t+b & t<4\\ 2-bt^2 & t\geq 4\end{cases}$, where b is a constant.

a. Find $\lim\limits_{t\to 4^+} f(t)$ and $\lim\limits_{t\to 4^-} f(t)$ in terms of b.

b. If $\lim\limits_{t\to 4} f(t)$ exists, determine the value of b.

Answers:

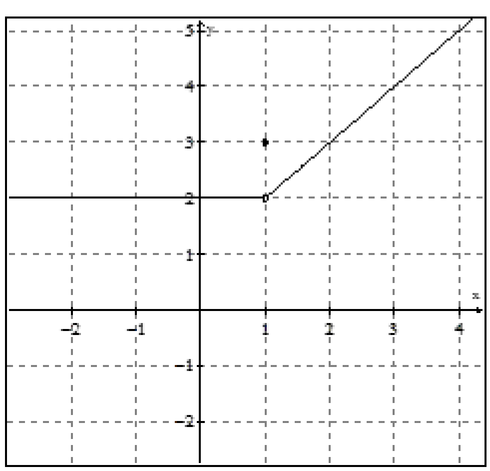
1. a) $\lim\limits_{x\to -3^-} f(x)=-2$ b) $\lim\limits_{x\to -3^+} f(x)=-2$ c) $\lim\limits_{x\to -3} f(x)=-2$

d) $\lim\limits_{x\to 1^+} f(x)=1$ e) $\lim\limits_{x\to 1^-} f(x)=1$ f) $\lim\limits_{x\to 1} f(x)=1$

g) $\lim\limits_{x\to 3^-} f(x)=-2$ h) $\lim\limits_{x\to 3^+} f(x)=1$ i) $\lim\limits_{x\to 3} f(x)$ *does not exist*

j) $f(-3)$ = undefined k) $f(1)$ = 1 l) $f(3)$ = −2

2. a)

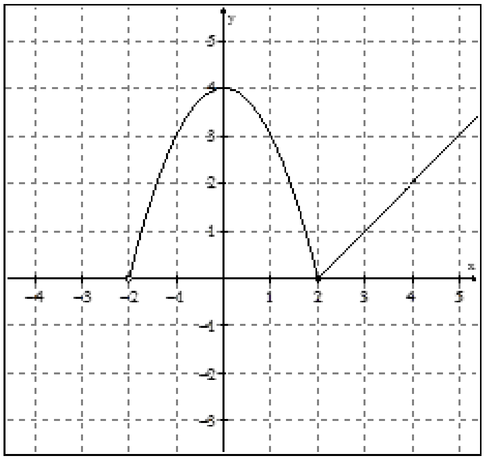


$\lim\limits_{x\to 1^-} f(x)=2$ and $\lim\limits_{x\to 1^+} f(x)=2$

Since the limits from the left and right of 1 are equal,

$\lim\limits_{x\to 1} f(x)=2$

b)

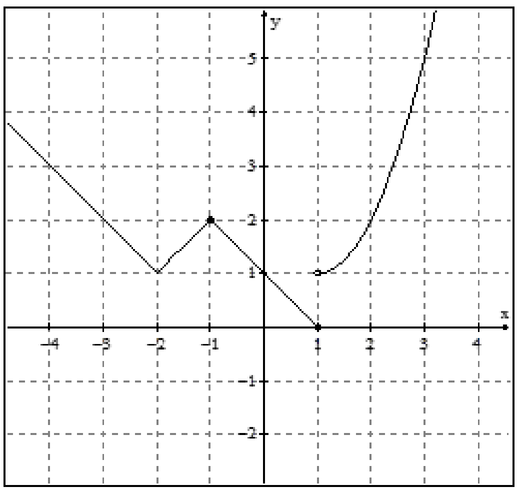


$\lim_{x \to 2^-} f(x) = 0$ *and* $\lim_{x \to 2^+} f(x) = 0$

Since the limits from the left and right of 2 are equal,

$\lim_{x \to 2} f(x) = 0$

c)



$\lim_{x \to 1^-} f(x) = 0$ and $\lim_{x \to 1^+} f(x) = 1$

Since the limits from the left and right of 1 are not equal,

$\lim_{x \to 1} f(x)$ does not exist.

3. a) $\lim_{x \to -2^+} f(x) = 4$ b) $\lim_{x \to 1^-} f(x) = 0$ c) $\textit{Since } \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = 4,$
$\textit{then } \lim_{x \to -1} f(x) = 4$

d)
$\lim_{x \to 0^-} f(x) = 3$
$\lim_{x \to 0^+} f(x) = 2$
$\textit{Since } \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x),$
$\textit{then } \lim_{x \to 0} f(x) \textit{ DNE}$

$\lim_{x \to 2^-} f(x) = 0$
$\lim_{x \to 2^+} f(x) = 0$
$\textit{Since } \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 0,$
$\textit{then } \lim_{x \to 0} f(x) = 0$

e)
$\lim_{x \to 2^-} f(x) = 0$
$\lim_{x \to 2^+} f(x) = 0$
$\textit{Since } \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 0,$
$\textit{then } \lim_{x \to 0} f(x) = 0$

4. $$f(x)=\begin{cases}\dfrac{3(-x+2)}{-x+2}, & x\le 2\\ \dfrac{3(x-2)}{-x+2}, & x>2\end{cases}=\begin{cases}3, & x\le 2\\ -3, & x>2\end{cases}$$

$a)\ \lim_{x\to 2^-} f(x)=3 \quad b)\ \lim_{x\to 2^+} f(x)=-3 \quad c)\ \textit{Since } \lim_{x\to 2^-} f(x)\ne \lim_{x\to 2^+} f(x), \textit{ then } \lim_{x\to 2} f(x)\ \textit{DNE}$

5. a) A possible defining equation is $f(x)=\begin{cases}2|x+1|-2 & x<0\\ 3 & 0\le x<2\\ -\sqrt{x-2}+3 & x>2\end{cases}$

b) i) The limit as x approaches 0 does not exist since the limit from the left does not equal the limit from the right.

$\lim_{x\to 0^-} f(x)=0 \qquad \lim_{x\to 0^+} f(x)=3 \qquad \text{Therefore } \lim_{x\to 0} f(x) \text{ does not exist}$

ii) The limit as x approaches 2 equals 3 since the limit from the left equals the limit from the right.

$\lim_{x\to 2^-} f(x)=3 \qquad \lim_{x\to 2^+} f(x)=3 \qquad \text{Therefore } \lim_{x\to 2} f(x)=3$

$6a.\quad \lim_{t\to 4^+} f(t)$

$= 2-16b \qquad \text{and} \qquad \lim_{t\to 4^-} f(t)=12+b$

b. If $\lim_{x\to 4} f(t)$ exists, then $\lim_{t\to 4^+} f(t)=\lim_{t\to 4^-} f(t)$, so $2-16b=12+b \ \to\ b=-\dfrac{10}{17}$