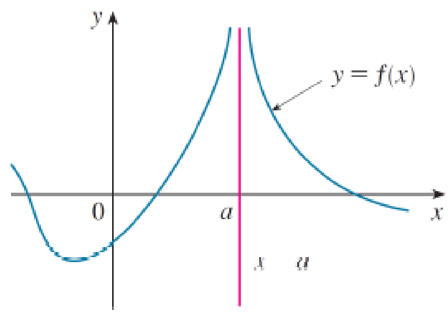
Infinite Limits

There are many examples of functions that increase and/or decrease without bound. These functions have limit behaviour that does not exist. If the y-values increase or decrease without bound as x approaches c, we can write $\lim_{x\to c} f(x) = +\infty$ or $\lim_{x\to c} f(x) = -\infty$, respectively. It is important to note that infinity is a behaviour, not a value. Stating that the limit of a given function is $+\infty$ or $-\infty$ is simply a *specific* way of saying that the limit does not exist.

Example 1:



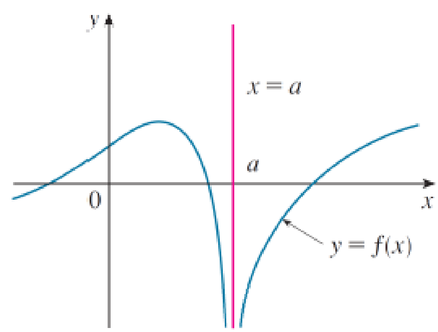
Find $\lim_{x\to a} f(x)$

Solution:

By looking at the graph of y=f(x), we can see that the y-values *increase* without bound as x approaches a from the left and from the right.

$\lim_{x\to a} f(x)$ = __________

Example 2:



Find $\lim_{x\to a} f(x)$

Solution:

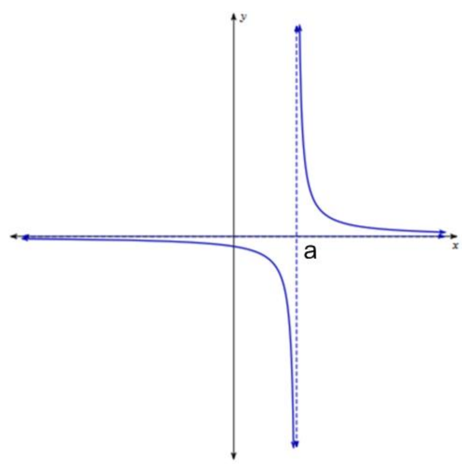
By looking at the graph of y=f(x), we can see that the y-values *decrease* without bound as x approaches a from the left and from the right.

$\lim_{x\to a} f(x)$ = __________

Example 3:

a. Find $\lim_{x\to a^-} f(x)$ b. Find $\lim_{x\to a^+} f(x)$

Solution:



a. By looking at the graph of y=f(x), we can see that the y-values *decrease* without bound as x approaches a from the *left*.

$\lim_{x\to a^-} f(x)$ = __________

b. By looking at the graph of y=f(x), we can see that the y-values *increase* without bound as x approaches a from the *right*.

$\lim_{x\to a^+} f(x)$ = __________

Vertical Asymptotes and Points of Discontinuity

To find these features of a given function, first factor the numerator and denominator. There will be either a vertical asymptote or a point of discontinuity at any x-value that would cause the denominator to equal zero.

Next, cancel like factors. If this eliminates the division by zero, then the function has a hole in the graph at the corresponding x-value. If canceling like factors does not eliminate the division by zero, then the function has a vertical asymptote at the corresponding x-value.

Once we have found the vertical asymptotes and points of discontinuity of a function, we can use limits to determine the behaviour of the function on each side of an asymptote and the y-coordinate of a point of discontinuity.

Functions that have *vertical asymptotes* approach positive or negative *infinity* as the x-values approach an asymptote from the left- and/or right-hand sides.

Functions that have *points of discontinuity* (holes) approach a *finite* value as the x-values approach a point of discontinuity from both sides.

Example 4:

Use factoring and simplifying to determine if there are any vertical asymptotes or points of discontinuity for the function $f(x)=\dfrac{x^2-3x+2}{x^2-4x+3}$.

Solution:

$$f(x)=\frac{x^2-3x+2}{x^2-4x+3}=$$

Since the factor $(x-1)$ cancels out of the denominator, we know there is a ________________ at x = _____.

Since the factor $(x-3)$ doesn't cancel out of the denominator, we know there is a ______________ at x = _____.

Example 5:

As discovered in the previous example, the function $f(x)=\dfrac{x^2-3x+2}{x^2-4x+3}$ has a point of discontinuity at x = 1.
Find the limit of f(x) as x approaches 1 in order to determine the y-coordinate of this point.

Solution:

$$\lim_{x\to 1}\frac{x^2-3x+2}{x^2-4x+3}=\lim_{x\to 1}\frac{(x-2)(x-1)}{(x-3)(x-1)}=\lim_{x\to 1}\frac{(x-2)}{(x-3)}=$$

Therefore, the coordinates of the P.O.D. are ________________.

Example 6:

As discovered in example 4, the function $f(x)=\dfrac{x^2-3x+2}{x^2-4x+3}$ has a vertical asymptote at x = 3. Find the limit of f(x) as x approaches 3 from the left and from the right in order to determine the behaviour of the function on each side of the asymptote.

Solution:

Substitute an x-value very close to 3, approaching from the left (x < 3), to calculate the following limit:

$$\lim_{x\to 3^-}\frac{x^2-3x+2}{x^2-4x+3}=\lim_{x\to 3^-}\frac{(x-2)(x-1)}{(x-3)(x-1)}=\lim_{x\to 3^-}\frac{(x-2)}{(x-3)}=$$

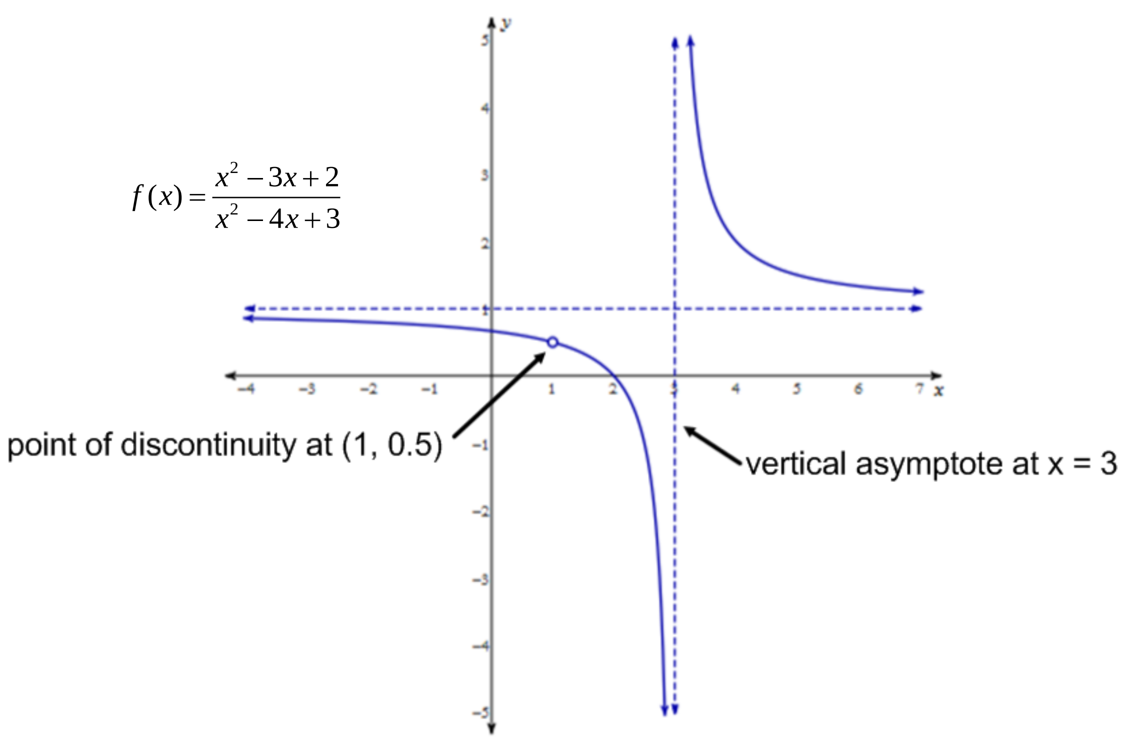
Therefore, the function ____________________ without bound as x approaches 3 from the left.

Substitute an x-value very close to 3, approaching from the right (x > 3), to calculate the following limit:

$$\lim_{x\to 3^+}\frac{x^2-3x+2}{x^2-4x+3}=\lim_{x\to 3^+}\frac{(x-2)(x-1)}{(x-3)(x-1)}=\lim_{x\to 3^+}\frac{(x-2)}{(x-3)}=$$

Therefore, the function ____________________ without bound as x approaches 3 from the right.

Note that the results of the previous examples are consistent with the features on the graph of f(x):



Limits and End Behaviour: Horizontal Asymptotes

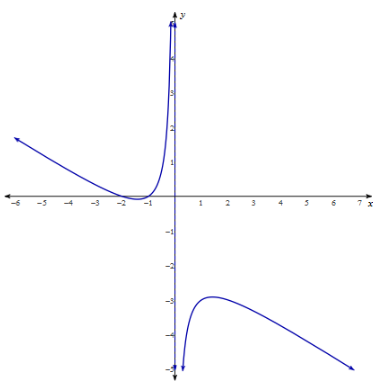
In order to determine the end behaviour of a function, we can evaluate limits at infinity. That is, we calculate a limit (algebraically, graphically, or numerically) as x approaches positive or negative infinity.

Using the result of the limit calculation, we can determine whether or not the graph of the function has a horizontal asymptote.

When direct substitution into the function produces the indeterminate form $\frac{\pm\infty}{\pm\infty}$ or $\infty-\infty$, we must be careful. These are cases which often have *finite* limits that can be determined through some algebraic manipulation.

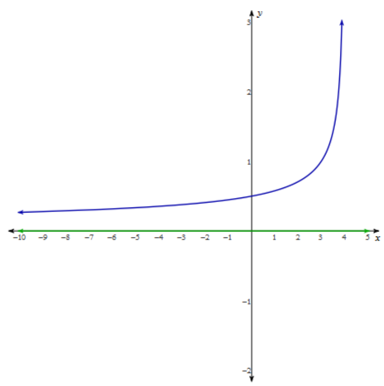
For any function, there are three possibilities for the *limit at infinity*:

Limit Does Not Exist



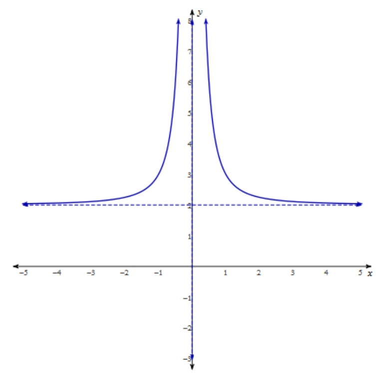
No horizontal asymptote

Limit is Zero



Horizontal asymptote at y = 0

Limit is a Non-Zero Number



Horizontal asymptote at y = limit
(In this case, y = 2)

Example 7: Determining a Limit at Infinity

Determine the end-behaviour of the function $f(x)=\dfrac{10x^2+7x}{3x^2-5x+1}$ by calculating $\lim\limits_{x\to\infty}\dfrac{10x^2+7x}{3x^2-5x+1}$.

Solution:

Direct substitution gives $\frac{\infty}{\infty}$.

Divide the numerator and denominator by the highest power of x, simplify, and then substitute.

$$\lim_{x\to\infty}\frac{10x^2+7x}{3x^2-5x+1}=$$

Therefore, there is a horizontal asymptote for the function $f(x)=\dfrac{10x^2+7x}{3x^2-5x+1}$ at __________.

Example 8: Determining a Limit at Infinity

Determine the end-behaviour of the function $f(x)=\dfrac{4x^3-6x^2+8}{7x^5+6x^2-9x+2}$ by calculating $\lim\limits_{x\to-\infty}\dfrac{4x^3-6x^2+8}{7x^5+6x^2-9x+2}$.

Solution:

Direct substitution gives $\dfrac{\infty}{\infty}$.

Divide the numerator and denominator by the highest power of x, simplify, and then substitute.

$$\lim_{x\to-\infty}\frac{4x^3-6x^2+8}{7x^5+6x^2-9x+2}=$$

Therefore, there is a horizontal asymptote for the function $f(x)=\dfrac{4x^3-6x^2+8}{7x^5+6x^2-9x+2}$ at __________.

Example 9: Determining a Limit at Infinity

Determine the end-behaviour of the function $f(x)=\dfrac{3-4x^2}{5x-1}$ by calculating $\lim\limits_{x\to\pm\infty}\dfrac{3-4x^2}{5x-1}$.

Solution:

Direct substitution gives $\dfrac{\infty}{\infty}$ or $\dfrac{-\infty}{\infty}$.

Divide the numerator and denominator by the highest power of x, simplify, and then substitute.

$$\lim_{x\to\pm\infty}\frac{3-4x^2}{5x-1}=$$

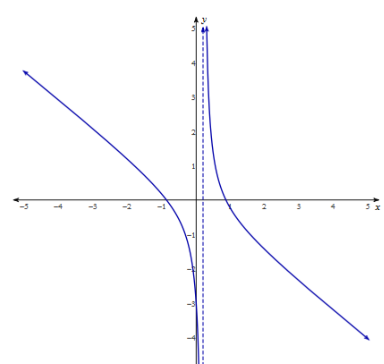
Therefore, there is ______________________________________ for the function $f(x)=\dfrac{3-4x^2}{5x-1}$.

To determine whether the function *increases* or *decreases* without bound, we can determine the limits at infinity separately (That is, the limit as x approaches *negative* infinity *and* the limit as x approaches *positive* infinity).

Substitute any representative x-value (very small, tending toward negative infinity, and then very large, tending toward positive infinity), into the *leading terms* of the numerator and denominator to determine if the overall limit will be *positive* or *negative* infinity.

$$\lim_{x\to-\infty}\frac{3-4x^2}{5x-1}=\frac{-4(\underline{\hspace{3cm}})^2}{5(\underline{\hspace{3cm}})}= \qquad \lim_{x\to+\infty}\frac{3-4x^2}{5x-1}=\frac{-4(\underline{\hspace{3cm}})^2}{5(\underline{\hspace{3cm}})}=$$

Note that these limits reflect the end behaviour of the graph of $f(x)=\dfrac{3-4x^2}{5x-1}$:



Example 10: Limits at Infinity Involving Radicals

Determine $\lim_{x\to+\infty} \frac{\sqrt{2x^3 - x}}{3x^2 - 6}$

Solution:

Direct substitution gives $\frac{\infty}{\infty}$.

Divide the numerator and denominator by the highest power of x, simplify, and then substitute.

Recall that, for example, $x^2 = \sqrt{x^4}$.

$$\lim_{x\to+\infty} \frac{\sqrt{2x^3 - x}}{3x^2 - 6} =$$

Example 11: Limits at Infinity Involving Radicals

Determine $\lim_{x\to+\infty} (\sqrt{x^2 - 4x} - x)$

Solution:

Direct substitution gives $\infty - \infty$.

Multiply by the conjugate form of 1. Then, divide the numerator and denominator by the highest power of x, simplify, and then substitute.

$$\lim_{x\to+\infty} (\sqrt{x^2 - 4x} - x) =$$

Practice:

1. Explain in your own words the meaning of each of the following:

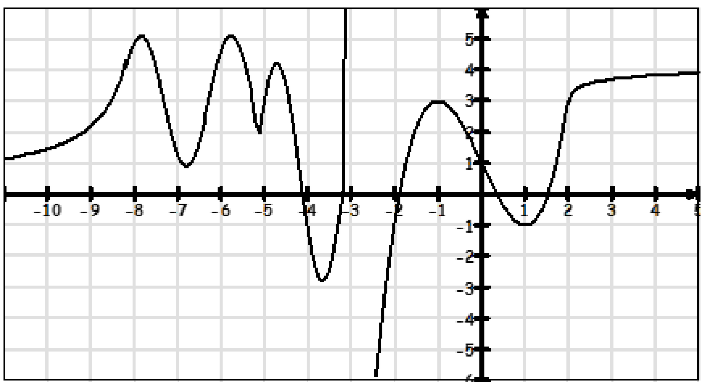
a) $\lim_{x \to -\infty} f(x) = 6$ b) $\lim_{x \to \infty} f(x) = -9$ c) $\lim_{x \to 4^+} f(x) = \infty$ d) $\lim_{x \to 6^-} f(x) = -\infty$

2. Can the graph of $y = f(x)$ intersect the following? Explain.
   a) vertical asymptote
   b) horizontal asymptote

3. For the function whose graph is shown below, determine the following:

a) $\lim_{x \to \infty} f(x)$ b) $\lim_{x \to -\infty} f(x)$ c) $\lim_{x \to -3^-} f(x)$ d) $\lim_{x \to -3^+} f(x)$

e) State the equations of the vertical and horizontal asymptotes.



4. Evaluate $\lim_{x \to \infty} \frac{x^2}{2^x}$ by using a table of values.

5. Evaluate the following limits.

a) $\lim_{x \to \infty} \frac{5x^4 - 7x^3 + 7x^2 - 1}{3x^4 + 2x^3}$ b) $\lim_{x \to -\infty} \frac{x^5 - x^2}{x^3 - 2x}$ c) $\lim_{x \to \infty} \frac{9x^2 - x + 8}{2x^4 + x^3 - 7}$

d) $\lim_{x \to -\infty} \frac{12x^2 - 6x^3 + 5x^4 + 9x^5}{3x^5 + 2x^4 - 4x^3 + 2x}$ e) $\lim_{x \to \infty} \frac{5x^3 - 4x^2 - 5x}{4x^3 + 3x}$ f) $\lim_{x \to \infty} [\left(\frac{1}{8}\right)^x + \frac{x^3 - 4x^2 - 5x}{4x^3 + 3x} - 7]$

g) $\lim_{x \to \infty} [\left(\frac{7}{3}\right)^{-x} + \frac{4x^2 - 5x}{2x^2 + 1} - 9]$ h) $\lim_{x \to -\infty} \left(\frac{1}{5}\right)^{2x}$ i) $\lim_{x \to \infty} \frac{(-2x^2 - 3)(x + 1)}{2 - 5x^3}$

j) $\lim_{x \to -\infty} [12 - \left(\frac{6}{5}\right)^x]$ k) $\lim_{x \to \infty} \frac{10x^2}{\sqrt{4x^4 + 1}}$ l) $\lim_{x \to \infty} \frac{2x}{\sqrt{x^2 - 2} - 3x}$

m) $\lim_{x \to \infty} \frac{4 - \frac{3}{x}}{5x^2 + 1}$ n) $\lim_{x \to \infty} [\frac{6x}{2x - 1} - \frac{x + 5}{3x - 4}]$ o) $\lim_{x \to \infty} [x - \sqrt{x^2 + 1}]$

p) $\lim_{x \to -\infty} \frac{|x - 8|}{x - 8}$ q) $\lim_{x \to \infty} [2x - \sqrt{4x^2 + 6x}]$ r) $\lim_{x \to -\infty} (x - 3)^2 (x + 1)^5$

s) $\lim_{x \to -\infty} (2x - 3)^3 (x + 1)^3$

6. Consider each of the following functions:

i. $f(x)=\dfrac{x^2+8x-20}{2x^2+x-6}$ ii. $f(x)=\dfrac{-2x+4}{x^3+4x^2-3x-18}$ iii. $f(x)=\dfrac{x^4-2x^3-63x^2}{x^2-10x+16}$

iv. $f(x)=\dfrac{10x^3-18x}{x^3-x^2-2x}$ v. $f(x)=\dfrac{9-x^2}{16-2x^3}$ vi. $f(x)=\dfrac{x^3+4x^2+3x}{x^2+8x+15}$

a. Determine if there are any vertical asymptotes and/or points of discontinuity, and for which values of x these occur.

b. ***Use limits*** to determine the behavior of f(x) on each side of any vertical asymptote that exists.

c. ***Use limits*** to determine the coordinates of any point of discontinuity that exists.

d. ***Use limits*** to determine whether or not a horizontal asymptote exists and, if so, what its equation is.

Answers:

1. a) As x approaches negative infinity, the value of the function approaches 6.
   b) As x approaches positive infinity, the value of the function approaches -9.
   c) As x approaches 4 from the right hand side, the value of the function gets increasingly large, approaching positive infinity.
   d) As x approaches 6 from the left hand side, the value of the function becomes more negative, approaching negative infinity.

2. a) No, the function is undefined at a vertical asymptote.
   b) Yes, the graph of a function can cross a horizontal asymptote. A horizontal asymptote describes the *end-behaviour* of a function - its tendency to approach a particular y-value as x approaches positive or negative infinity.

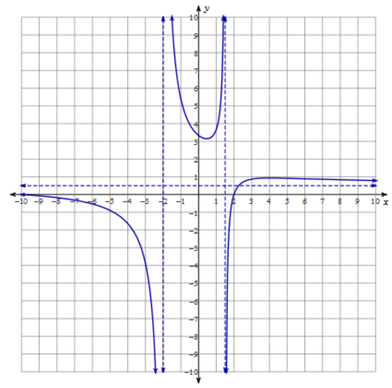
3. a) 4
   b) 1
   c) ∞
   d) $-\infty$
   e) $HA\ :\ y = 1\ and\ y = 4$
   $VA\ :\ x = -3$

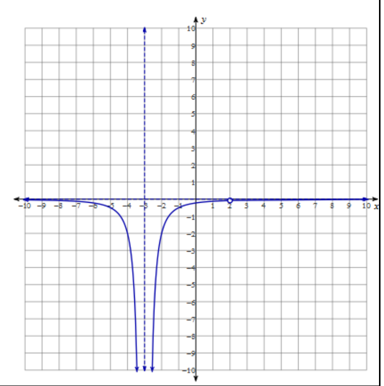
4. $\lim\limits_{x\to\infty} \dfrac{x^2}{2^x} = 0$

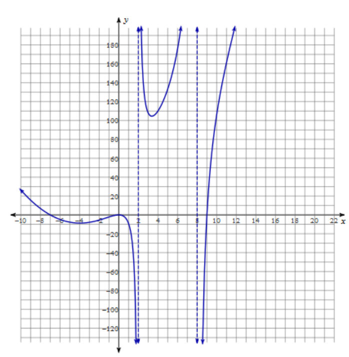
5. a) $\frac{5}{3}$ b) ∞ c) 0 d) 3 e) $\frac{5}{4}$ f) $-\frac{27}{4}$ g) -7 h) ∞ i) $\frac{2}{5}$ j) 12

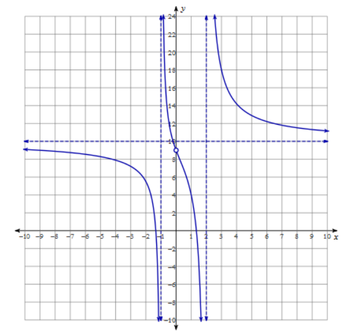
   k) 5 l) -1 m) 0 n) $\frac{8}{3}$ o) 0 p) -1 q) $-\frac{3}{2}$ r) $-\infty$ s) $\propto$

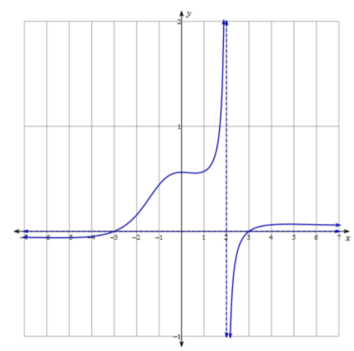
6.

i) $f(x)=\dfrac{x^2+8x-20}{2x^2+x-6}$	VA x=3/2 x=–2	$\lim\limits_{x\to -2^-} f(x)=-\infty$ $\lim\limits_{x\to -2^+} f(x)=+\infty$ $\lim\limits_{x\to \frac{3}{2}^-} f(x)=+\infty$ $\lim\limits_{x\to \frac{3}{2}^+} f(x)=-\infty$	No POD	$\lim\limits_{x\to\infty} f(x)=\dfrac{1}{2}$ HA $y=\dfrac{1}{2}$	

ii) $f(x)=\dfrac{-2x+4}{x^3+4x^2-3x-18}$	VA x=–3 POD at x=2	$\lim\limits_{x\to -3^-} f(x)=-\infty$ $\lim\limits_{x\to -3^+} f(x)=-\infty$	POD (2, –0.08)	$\lim\limits_{x\to\infty} f(x)=0$ HA $y=0$	

iii) $f(x)=\dfrac{x^4-2x^3-63x^2}{x^2-10x+16}$	VA x=2 x=8	$\lim\limits_{x\to 2^-} f(x)=-\infty$ $\lim\limits_{x\to 2^+} f(x)=+\infty$ $\lim\limits_{x\to 8^-} f(x)=+\infty$ $\lim\limits_{x\to 8^+} f(x)=-\infty$	No POD	$\lim\limits_{x\to\infty} f(x)=DNE$ No HA	

iv) $f(x)=\dfrac{10x^3-18x}{x^3-x^2-2x}$	VA x=–1 x=2	$\lim_{x\to -1^-} f(x)=-\infty$ $\lim_{x\to -1^+} f(x)=+\infty$ $\lim_{x\to 2^-} f(x)=-\infty$ $\lim_{x\to 2^+} f(x)=+\infty$	POD (0, 9)	$\lim_{x\to\infty} f(x)=10$ HA y=10	

v) $f(x)=\dfrac{9-x^2}{16-2x^3}$	VA x=2	$\lim_{x\to 2^-} f(x)=+\infty$ $\lim_{x\to 2^+} f(x)=-\infty$	No POD	$\lim_{x\to\infty} f(x)=0$ HA y=0	

vi) $f(x)=\dfrac{x^3+4x^2+3x}{x^2+8x+15}$	VA x=–5 POD at x=–3	$\lim_{x\to -5^-} f(x)=-\infty$ $\lim_{x\to -5^+} f(x)=+\infty$	POD (–3, 3)	$\lim_{x\to\infty} f(x)=DNE$ No HA