

# Understanding Limits

In algebra, we are often required to calculate an exact value of a function  $y=f(x)$ . When determining a specific y-value of a function, we simply need the corresponding x-value.

However, what if the function isn't just a simple curve? For example, we will be examining functions with holes, functions that shoot up toward infinity, functions that oscillate, etc. To describe the behaviour of the y-values of such functions, we can use a *limit*.

## Intuitive Definition of a Limit

The limit of a function describes how a function behaves *near* a specific point, but not *at* that point. If the values of  $y=f(x)$  get closer and closer to one number, L, as we take values of x very close to (but not equal to) a number, c, then we say "**The limit of f(x), as x approaches c, is L**" and we write:

$$\lim_{x \rightarrow c} f(x) = L$$

It is important to note that the value of the *limit* at  $x = c$  does *not* depend on the value of the *function* at  $x = c$ .

- $f(c)$  is a single number that describes the value of  $f(x)$  *at* point  $x = c$ .
- $\lim_{x \rightarrow c} f(x)$  is a single number that describes the behavior of  $f(x)$  *near, but not at*, the point  $x = c$ .

## Using Tables to Find Limits

One way to approximate a limit is to use a table of values. Choose x-values close to c from both the left and the right. If the y-values approach a distinct value, L, as x approaches c, then the limit is L.

### Example 1:

Determine the value of  $\lim_{x \rightarrow 5} \frac{x^2 - 3x - 10}{x - 5}$

#### Solution:

- Create a table of values for the function  $f(x) = \frac{x^2 - 3x - 10}{x - 5}$  and fill in values of x that get close to 5 from both sides. Then, calculate the corresponding f(x) values.

x	4	4.5	4.9	4.95	5	5.05	5.1	5.5	6
f(x)									

- Examine the table to determine if the f(x) values approach the same number from the left and the right.

$$\lim_{x \rightarrow 5} \frac{x^2 - 3x - 10}{x - 5} = \underline{\hspace{2cm}}$$

\* Note that we cannot find the limit by substituting  $x = 5$  into the function since this results in division by zero.

# Using Graphs to Find Limits

If we have the graph of a function  $f(x)$  near  $x = c$ , then it is usually easy to determine  $\lim_{x \rightarrow c} f(x)$ .

## Example 2:

Use the graph of the function to the right to determine the following limits.

- a.  $\lim_{x \rightarrow 4} f(x)$       b.  $\lim_{x \rightarrow 2} f(x)$

### Solution:

- a. As the  $x$ -values get closer to 4 from the left and from the right (We will learn more about Left- & Right-Hand Limits later), the  $y$ -values get closer to \_\_\_\_.

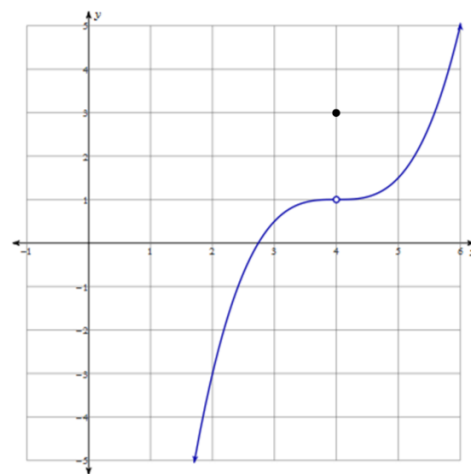
Note that  $f(4) = \underline{\hspace{2cm}}$ , but the limit of  $f(x)$  as  $x$  approaches 4 is \_\_\_\_.

$$\lim_{x \rightarrow 4} f(x) = \underline{\hspace{2cm}}$$

- b. As the  $x$ -values get closer to 2 from the left and from the right, the  $y$ -values get closer to \_\_\_\_.

Note that  $f(2) = \underline{\hspace{2cm}}$ . In this case, the function value and the limit value are the same.

$$\lim_{x \rightarrow 2} f(x) = \underline{\hspace{2cm}}$$



## Example 3:

Use the graph of the function to the right to determine the following limits.

- a.  $\lim_{x \rightarrow a} f(x)$       b.  $\lim_{x \rightarrow b} f(x)$       c.  $\lim_{x \rightarrow c} f(x)$

### Solution:

- a. As the  $x$ -values get closer to  $a$  from either side, the  $y$ -values get closer to \_\_\_\_.

Note that the function doesn't exist at  $a$ , but the limit does.

$$\lim_{x \rightarrow a} f(x) = \underline{\hspace{2cm}}$$

- b. As the  $x$ -values get closer to  $b$  from the left, the  $y$ -values get closer to \_\_\_\_.  
As the  $x$ -values get closer to  $b$  from the right, the  $y$ -values get closer to \_\_\_\_.

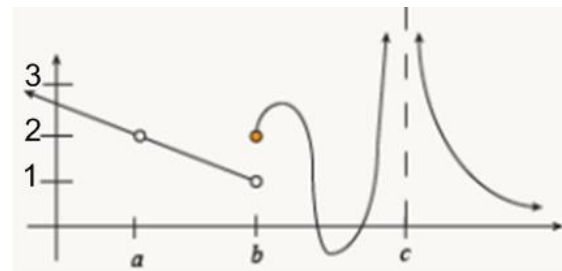
Since the function approaches different values from each side, the limit *does not exist*.

$$\lim_{x \rightarrow b} f(x) \underline{\hspace{2cm}}$$

- c. As the  $x$ -values get closer to  $c$  from either side, the  $y$ -values get closer to \_\_\_\_.

Since the function approaches infinity from each side, the limit *does not exist*.

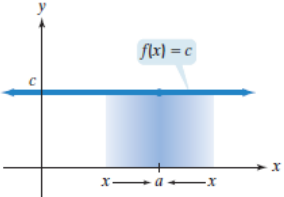
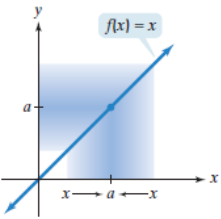
$$\lim_{x \rightarrow c} f(x) \underline{\hspace{2cm}}$$



# Calculating Limits Algebraically

## Limit Rules

If  $n$  is a positive integer,  $c$  is a constant, and  $f(x)$  and  $g(x)$  are functions that have limits at  $x = a$ , then the following rules hold:

<p>1. <b>Constant Rule</b></p> $\lim_{x \rightarrow a} c = c$		<p>6. <b>Product Rule</b></p> $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
<p>2. <b>Identity Rule</b></p> $\lim_{x \rightarrow a} x = a$		<p>7. <b>Quotient Rule</b></p> $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \text{ if } \lim_{x \rightarrow a} g(x) \neq 0$
<p>3. <b>Constant Multiple Rule</b></p> $\lim_{x \rightarrow a} c f(x) = c \lim_{x \rightarrow a} f(x)$		<p>8. <b>Power Rule</b></p> $\lim_{x \rightarrow a} [f(x)]^n = \left[ \lim_{x \rightarrow a} f(x) \right]^n$
<p>4. <b>Sum Rule</b></p> $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$		<p>9. <b>n<sup>th</sup> Root Rule</b></p> $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}, \text{ if } \lim_{x \rightarrow a} f(x) > 0$
<p>5. <b>Difference Rule</b></p> $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$		

## Example 4: Finding a Limit Using Limit Rules

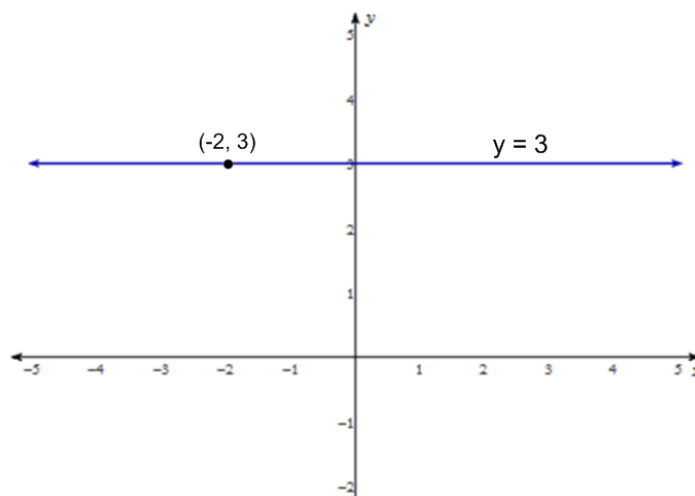
Find  $\lim_{x \rightarrow -2} 3$

**Solution:**

Use the Constant Rule for limits.

$$\lim_{x \rightarrow -2} 3 = \underline{\hspace{2cm}}$$

Look at the graph of the function  $y = 3$  to see why the limit of the constant function as  $x$  approaches 3 is the same as the constant value. Notice that for this function, the limit as  $x$  approaches *any* value is 3.



**Example 5: Finding a Limit Using Limit Rules**

Find  $\lim_{x \rightarrow 3} (2x^2 - 4x - 1)$

**Solution:**

Use the Difference Rule:

Use the Constant Multiple Rule:

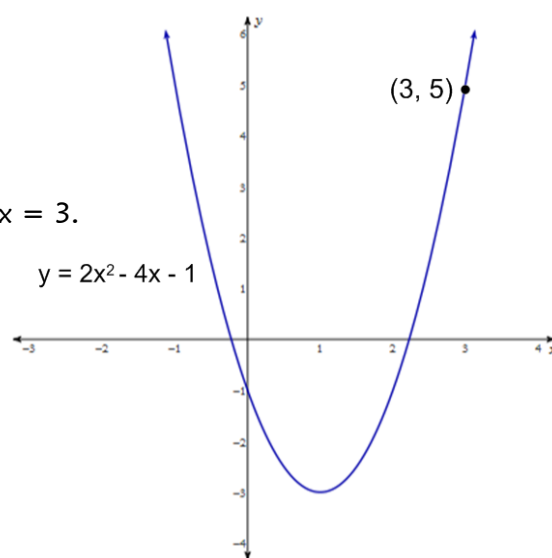
Use the Power Rule:

Use the Constant Rule:

Substitute &amp; Simplify:

Note that the limit as  $x$  approaches 3 is the same as the function value at  $x = 3$ .

This is always the case for limits of polynomial functions.

We can use *direct substitution* to evaluate all polynomial limits.**Example 6: Finding a Limit by Direct Substitution**

a. Find  $\lim_{x \rightarrow -1} (-x^4 + 5x^3 + 3x^2 - 1)$       b. Find  $\lim_{x \rightarrow 8} \frac{x^2 - 2x - 3}{x + 2}$

**Solution:**

a. Use direct substitution.

$$\lim_{x \rightarrow -1} (-x^4 + 5x^3 + 3x^2 - 1) =$$

b. Use direct substitution.

$$\lim_{x \rightarrow 8} \frac{x^2 - 2x - 3}{x + 2} =$$

**Example 7: Finding a Limit by Factoring & Simplifying First**

a. Find  $\lim_{x \rightarrow 1} \frac{2x^2 - x - 1}{x - 1}$

b. Find  $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2}$

**Solution:**

a. If we try to evaluate  $\lim_{x \rightarrow 1} \frac{2x^2 - x - 1}{x - 1}$  by using direct substitution, we get  $\frac{2(1)^2 - 1 - 1}{1 - 1} = \frac{0}{0}$ .

So, factor and simplify first, *then* use substitution.

$$\lim_{x \rightarrow 1} \frac{2x^2 - x - 1}{x - 1} =$$

b. If we try to evaluate  $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2}$  by using direct substitution, we get  $\frac{(2)^3 - 8}{2 - 2} = \frac{0}{0}$ .

So, factor and simplify first, *then* use substitution.

$$\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2} =$$

**Example 8: Finding a Limit by Using the Conjugate Technique**

a. Find  $\lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x}-2}$

b. Find  $\lim_{x \rightarrow -3} \frac{2-\sqrt{x^2-5}}{x+3}$

c. Find  $\lim_{x \rightarrow 0} \frac{\sqrt[3]{x+27}-3}{x}$

**Solution:**a. Direct substitution yields  $\frac{0}{0}$ . So, multiply by the conjugate form of 1.

$$\lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x}-2} =$$

b. Direct substitution yields  $\frac{0}{0}$ . So, multiply by the conjugate form of 1.

$$\lim_{x \rightarrow -3} \frac{2-\sqrt{x^2-5}}{x+3} =$$

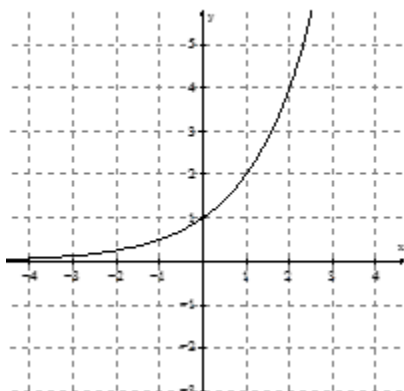
- c. Direct substitution yields  $\frac{0}{0}$ . So, multiply by the conjugate form of 1.

$$\lim_{x \rightarrow 0} \frac{\sqrt[3]{x+27} - 3}{x} =$$

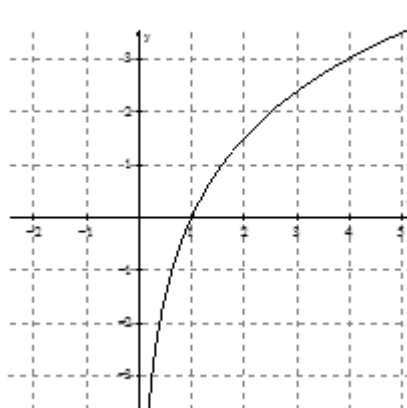
## Practice:

1. Determine the following limits graphically.

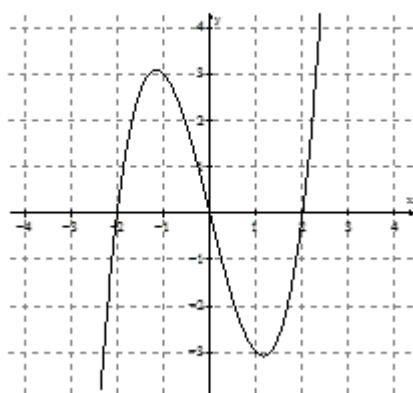
a.  $\lim_{x \rightarrow 1} f(x)$



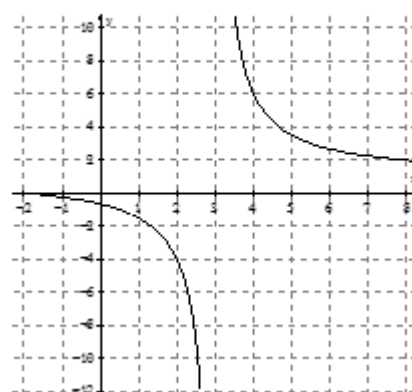
b.  $\lim_{x \rightarrow 1} f(x)$



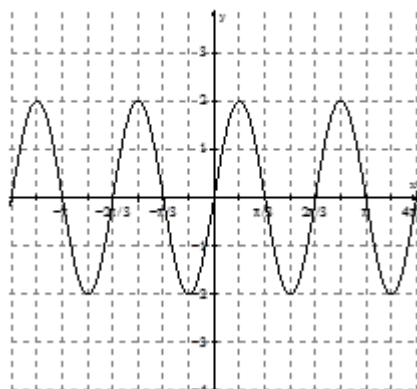
c.  $\lim_{x \rightarrow -1} f(x)$



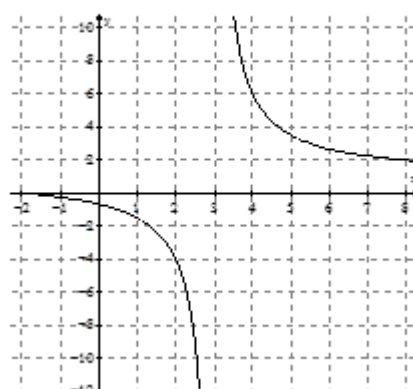
d.  $\lim_{x \rightarrow 3} f(x)$



e.  $\lim_{x \rightarrow \frac{\pi}{6}} f(x)$



f.  $\lim_{x \rightarrow 4} f(x)$

2. Use the *Limit Rules* to calculate  $\lim_{x \rightarrow 3} \frac{(4x+2)^2}{\sqrt[3]{1-x^2}}$ .



3. Evaluate the following limits.

a)  $\lim_{x \rightarrow 4} \frac{x-4}{x^3-64}$

j)  $\lim_{x \rightarrow -8} \frac{x+8}{\sqrt[3]{x}+2}$

s)  $\lim_{x \rightarrow 5} \frac{2x}{x-5}$

b)  $\lim_{x \rightarrow 7} \frac{\sqrt[5]{3-5x}}{(x-5)^3}$

k)  $\lim_{x \rightarrow -1} \frac{8}{(3+x)\sqrt{3-x}}$

t)  $\lim_{x \rightarrow 0} \frac{x^3}{2x^3+3x^4}$

c)  $\lim_{x \rightarrow -3} \frac{x+3}{3-\sqrt{x+12}}$

l)  $\lim_{x \rightarrow 9} \frac{x-9}{3-\sqrt{x}}$

u)  $\lim_{x \rightarrow 2} \frac{x^2-2x}{2x^2-7x+6}$

d)  $\lim_{x \rightarrow -3} \sqrt[3]{\frac{x-4}{6x^2+2}}$

m)  $\lim_{x \rightarrow 0} \frac{(3+x)^3-3^3}{(3+x)^2-3^2}$

v)  $\lim_{x \rightarrow \pi} \frac{1+\sin x}{\cos x}$

e)  $\lim_{x \rightarrow 3} \frac{\frac{1}{x^2}-\frac{1}{9}}{x-3}$

n)  $\lim_{x \rightarrow 9} \frac{x^2-81}{3-\sqrt{x}}$

w)  $\lim_{x \rightarrow \frac{\pi}{4}} ((\cos x) + (\sin x))$

f)  $\lim_{h \rightarrow 0} \frac{(-3+h)^2-9}{h}$

o)  $\lim_{x \rightarrow 16} \frac{2\sqrt{x}+x^{3/2}}{\sqrt[4]{x}+5}$

x)  $\lim_{x \rightarrow \frac{3\pi}{2}} \frac{\sin x}{\cos x}$

g)  $\lim_{h \rightarrow 0} \frac{(2+h)^3-8}{h}$

p)  $\lim_{x \rightarrow 2} \frac{x^2-4}{3x^2-7x+2}$

y)  $\lim_{x \rightarrow 4} \frac{2x^3-128}{\sqrt{x}-2}$

h)  $\lim_{x \rightarrow 0} \frac{\frac{1}{1+x}-1}{x}$

q)  $\lim_{x \rightarrow 7} \frac{\sqrt{x+2}-3}{x-7}$

z)  $\lim_{x \rightarrow 0} \frac{\sqrt[3]{h+1}-1}{h}$

i)  $\lim_{x \rightarrow -1} \frac{2x+3}{3x+2}$

r)  $\lim_{x \rightarrow 4} \left( \frac{8}{x^2-16} - \frac{1}{x-4} \right)$

## Answers:

1. a. 2   b. 0   c. 3   d. DNE   e. 2   f. 6

2.

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{(4x+2)^2}{\sqrt[3]{1-x^2}} &= \frac{\lim_{x \rightarrow 3} (4x+2)^2}{\lim_{x \rightarrow 3} \sqrt[3]{1-x^2}} = \frac{\left( \lim_{x \rightarrow 3} (4x+2) \right)^2}{\sqrt[3]{\lim_{x \rightarrow 3} (1-x^2)}} = \frac{\left( \lim_{x \rightarrow 3} (4x) + \lim_{x \rightarrow 3} 2 \right)^2}{\sqrt[3]{\lim_{x \rightarrow 3} 1 - \lim_{x \rightarrow 3} (x^2)}} = \frac{\left( 4 \lim_{x \rightarrow 3} x + \lim_{x \rightarrow 3} 2 \right)^2}{\sqrt[3]{\lim_{x \rightarrow 3} 1 - \left( \lim_{x \rightarrow 3} x \right)^2}} \\ &= \frac{(4(3)+2)^2}{\sqrt[3]{1-(3)^2}} \\ &= \frac{196}{\sqrt[3]{-8}} = \frac{196}{-2} = -98 \end{aligned}$$

3. a)  $\frac{1}{48}$    b)  $-\frac{1}{4}$    c) -6   d)  $-\frac{1}{2}$    e)  $-\frac{2}{27}$    f) -6   g) 12   h) -1i) -1   j) 12   k) 2   l) -6   m)  $\frac{9}{2}$    n) -108   o)  $\frac{72}{7}$    p)  $\frac{4}{5}$    q)  $\frac{1}{6}$ r)  $-\frac{1}{8}$    s) DNE   t)  $\frac{1}{2}$    u) 2   v) -1   w)  $\sqrt{2}$    x) DNE   y) 384   z)  $\frac{1}{3}$