

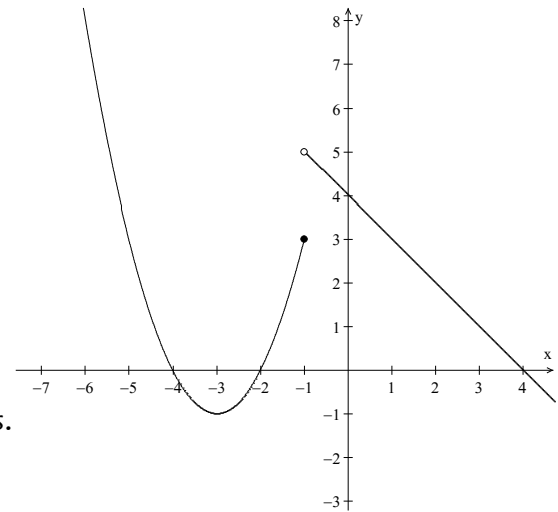
# Left-Hand and Right-Hand Limits

The graph of following the piecewise-defined function shows that there is a break in the graph.

$$f(x) = \begin{cases} x^2 + 6x + 8 & x \leq -1 \\ -x + 4 & x > -1 \end{cases}$$

Notice that as  $x$  approaches  $-1$  *from the left*, the  $f(x)$  values are approaching \_\_\_\_\_, but as  $x$  approaches  $-1$  *from the right*, the  $f(x)$  values are approaching \_\_\_\_\_. Since the  $f(x)$  values do not approach the *same* number as  $x$  approaches  $-1$  from the left and from the right, the limit *does not exist* at  $x = -1$ .

However, even though the limit does not exist at  $x = -1$ , we can still describe the behaviour of the graph using one-sided limits. These one-sided limits have their own special notation.



**Left-Hand Limit Notation:**  $\lim_{x \rightarrow c^-} f(x)$  “the limit of  $f(x)$  as  $x$  approaches  $c$  from the left”

**Right-Hand Limit Notation:**  $\lim_{x \rightarrow c^+} f(x)$  “the limit of  $f(x)$  as  $x$  approaches  $c$  from the right”

## One-Sided Limit Theorem

$$\lim_{x \rightarrow c} f(x) = L \text{ if and only if } \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L$$

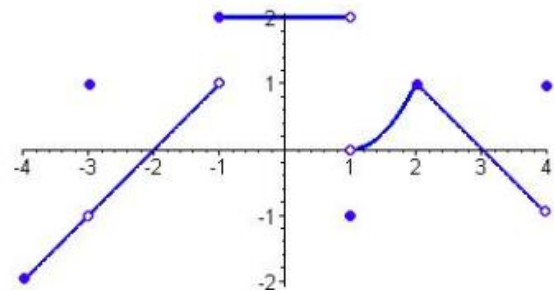
We can use this theorem in two different ways:

1. If  $\lim_{x \rightarrow c^-} f(x) \neq \lim_{x \rightarrow c^+} f(x)$ , then  $\lim_{x \rightarrow c} f(x)$  does not exist.
2. If  $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L$ , then  $\lim_{x \rightarrow c} f(x) = L$

## Example 1: Finding One-Sided Limits from a Graph

Find the following limits, if they exist.

a.  $\lim_{x \rightarrow -3} f(x)$       b.  $\lim_{x \rightarrow 1} f(x)$



**Solution:**

Use the graph of  $f(x)$  shown above. For the limit to exist, the left-and right-hand limits must equal each other.

a.  $\lim_{x \rightarrow -3^-} f(x) = \underline{\hspace{2cm}}$        $\lim_{x \rightarrow -3^+} f(x) = \underline{\hspace{2cm}}$       Since  $\lim_{x \rightarrow -3^-} f(x) = \lim_{x \rightarrow -3^+} f(x)$ , then  $\lim_{x \rightarrow -3} f(x) = \underline{\hspace{2cm}}$

b.  $\lim_{x \rightarrow 1^-} f(x) = \underline{\hspace{2cm}}$        $\lim_{x \rightarrow 1^+} f(x) = \underline{\hspace{2cm}}$       Since  $\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$ , then  $\lim_{x \rightarrow 1} f(x) = \underline{\hspace{2cm}}$

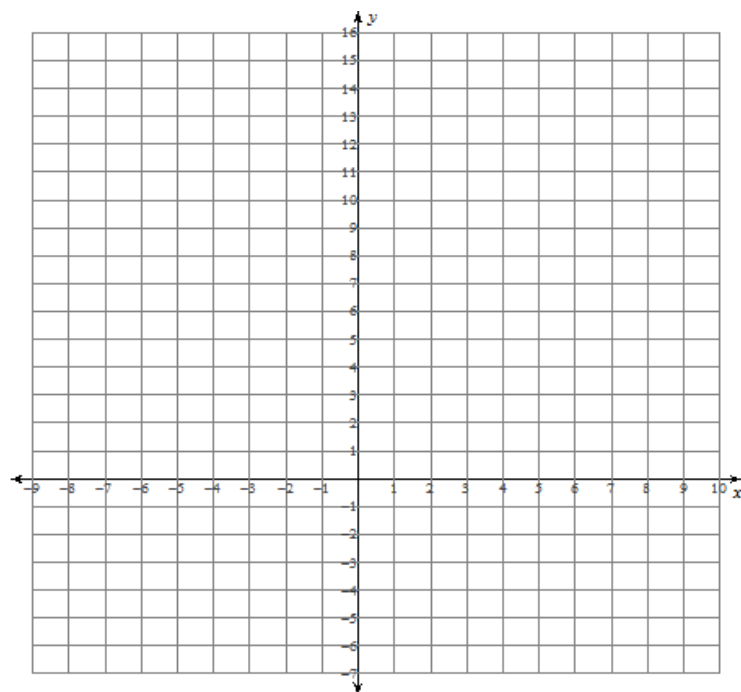
## Example 2: Sketching Graphs of Piecewise Functions and Determining One-Sided Limits

a. Given the piecewise function:  $f(x) = \begin{cases} x^2 + 5, & \text{if } x < 2 \\ 3x + 1, & \text{if } x \geq 2 \end{cases}$

i. Sketch the graph of  $f(x)$ .

ii. Find the following limits, if they exist.

- $\lim_{x \rightarrow 2^-} f(x) = \underline{\hspace{2cm}}$
- $\lim_{x \rightarrow 2^+} f(x) = \underline{\hspace{2cm}}$
- $\lim_{x \rightarrow 2} f(x) = \underline{\hspace{2cm}}$

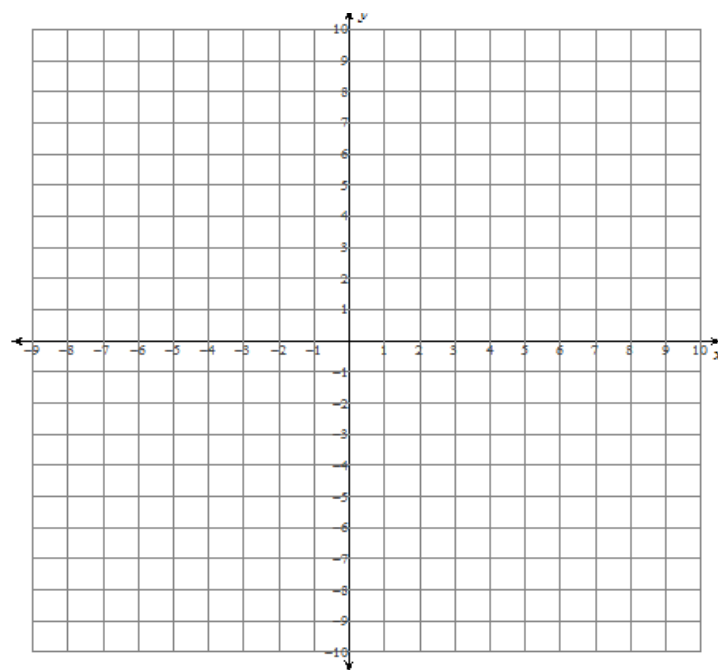


b. Given the piecewise function:  $f(x) = \begin{cases} -x^2 - 4x - 2, & x \leq 0 \\ (x-1)^3 - 1, & x > 0 \end{cases}$

i. Sketch the graph of  $f(x)$ .

ii. Find the following limits, if they exist.

- $\lim_{x \rightarrow 0^-} f(x) = \underline{\hspace{2cm}}$
- $\lim_{x \rightarrow 0^+} f(x) = \underline{\hspace{2cm}}$
- $\lim_{x \rightarrow 0} f(x) = \underline{\hspace{2cm}}$



**Example 3: Finding One-Sided Limits Given the Equation of a Function**

Given the piecewise function  $f(x) = \begin{cases} 2x - 1, & x \leq -1 \\ x^2 + 1, & -1 < x \leq 1 \\ -x + 3, & x > 1 \end{cases}$ , determine the following limits, if they exist.

a.  $\lim_{x \rightarrow -1} f(x)$       b.  $\lim_{x \rightarrow 1} f(x)$

**Solution:**

a. Step 1: Calculate the left-hand limit

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} (\text{_____}) =$$

Step 2: Calculate the right-hand limit

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (\text{_____}) =$$

Step 3: Compare the left- and right-hand limits

Since  $\lim_{x \rightarrow -1^-} f(x) \neq \lim_{x \rightarrow -1^+} f(x)$ , then  $\lim_{x \rightarrow -1} f(x) = \text{_____}$

b. Step 1: Calculate the left-hand limit

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (\text{_____}) =$$

Step 2: Calculate the right-hand limit

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (\text{_____}) =$$

Step 3: Compare the left- and right-hand limits

$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x)$ , so  $\lim_{x \rightarrow 1} f(x) = \text{_____}$

**Example 4: Finding One-Sided Limits Given the Equation of a Function**

Rewrite  $f(x) = \frac{2|x-6|}{x-6}$  as a piecewise function, and then determine  $\lim_{x \rightarrow 6} f(x)$ , if it exists.

**Solution:**

Recall that  $|x-6| = \begin{cases} x-6, & x \geq 6 \\ -x+6, & x < 6 \end{cases}$

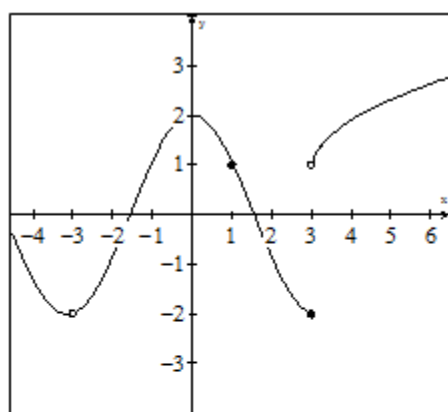
So,  $f(x) = \frac{2|x-6|}{x-6} =$

$\lim_{x \rightarrow 6^-} f(x) = \text{_____}$        $\lim_{x \rightarrow 6^+} f(x) = \text{_____}$

Since  $\lim_{x \rightarrow 6^-} f(x) \neq \lim_{x \rightarrow 6^+} f(x)$ , then  $\lim_{x \rightarrow 6} f(x) = \text{_____}$

## Practice:

1. Use the graph shown to determine the following limits.



- a)  $\lim_{x \rightarrow -3^-} f(x)$       b)  $\lim_{x \rightarrow -3^+} f(x)$       c)  $\lim_{x \rightarrow -3} f(x)$   
 d)  $\lim_{x \rightarrow 1^+} f(x)$       e)  $\lim_{x \rightarrow 1^-} f(x)$       f)  $\lim_{x \rightarrow 1} f(x)$   
 g)  $\lim_{x \rightarrow 3^-} f(x)$       h)  $\lim_{x \rightarrow 3^+} f(x)$       i)  $\lim_{x \rightarrow 3} f(x)$   
 j)  $f(-3)$       k)  $f(1)$       l)  $f(3)$

2. Sketch each piecewise function. Determine the given limit, if it exists. If the limit does not exist, provide an explanation.

a.  $f(x) = \begin{cases} 2 & , \quad x < 1 \\ 3 & , \quad x = 1 \\ x + 1 & , \quad x > 1 \end{cases}$       b.  $f(x) = \begin{cases} 4 - x^2 & , \quad -2 < x \leq 2 \\ x - 2 & , \quad x > 2 \end{cases}$       c.  $f(x) = \begin{cases} |x + 2| + 1 & , \quad x < -1 \\ -x + 1 & , \quad -1 \leq x \leq 1 \\ x^2 - 2x + 2 & , \quad x > 1 \end{cases}$

Find  $\lim_{x \rightarrow 1} f(x)$       Find  $\lim_{x \rightarrow 2} f(x)$       Find  $\lim_{x \rightarrow 1} f(x)$

3. Determine the given limit, if it exists. If the limit does not exist, provide an explanation.

a)  $f(x) = \begin{cases} 2x - 1, & x \leq -2 \\ -x + 2, & x > -2 \end{cases}$       b)  $f(x) = \begin{cases} -x^2 + 4x - 3, & x < 1 \\ x - 7, & x \geq 1 \end{cases}$       c)  $f(x) = \begin{cases} x^2 - 2x + 1, & x < -1 \\ -\frac{x}{2} + \frac{7}{2}, & x \geq -1 \end{cases}$

Find  $\lim_{x \rightarrow -2} f(x)$       Find  $\lim_{x \rightarrow 1} f(x)$       Find  $\lim_{x \rightarrow -1} f(x)$

d)  $f(x) = \begin{cases} x + 3, & x \in (-\infty, 0] \\ -x + 2, & x \in (0, 2) \\ (x - 2)^2, & x \in [2, \infty) \end{cases}$       e)  $f(x) = \begin{cases} (x + 1)^2 - 1, & -2 \leq x < 0 \\ \frac{5}{4} \sin\left(\frac{\pi x}{2}\right), & 0 \leq x < 2 \\ (x - 3)^2 - 1, & 2 \leq x \leq 4 \end{cases}$

Find  $\lim_{x \rightarrow 0} f(x)$  and  $\lim_{x \rightarrow 2} f(x)$       Find  $\lim_{x \rightarrow 2} f(x)$

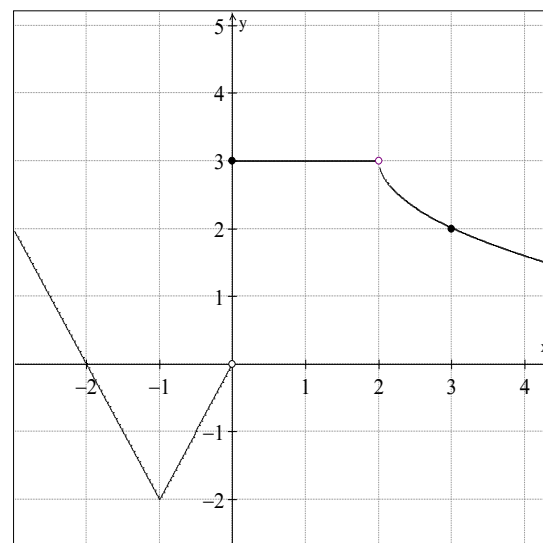
4. Rewrite  $f(x) = \frac{3|-x+2|}{-x+2}$  as a piecewise function and then determine the following limits.

a)  $\lim_{x \rightarrow 2^-} f(x)$     b)  $\lim_{x \rightarrow 2^+} f(x)$     c)  $\lim_{x \rightarrow 2} f(x)$

5. Consider the graph of a piecewise function shown to the right.

a. What is a possible equation for this function?

- b. i) Determine  $\lim_{x \rightarrow 0} f(x)$ , if it exists.  
 ii) Determine  $\lim_{x \rightarrow 2} f(x)$ , if it exists.



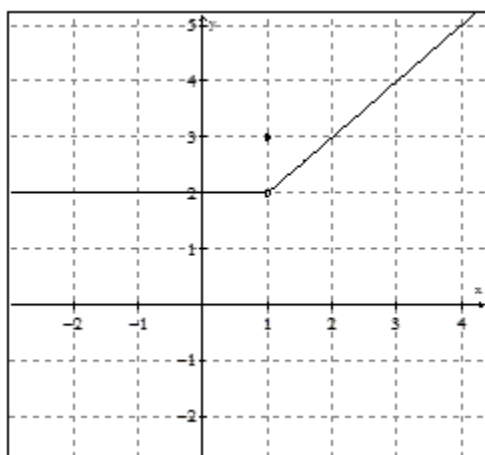
6. The function  $f(t)$  is defined by  $f(t) = \begin{cases} 3t + b & t < 4 \\ 2 - bt^2 & t \geq 4 \end{cases}$ , where  $b$  is a constant.

- a. Find  $\lim_{t \rightarrow 4^+} f(t)$  and  $\lim_{t \rightarrow 4^-} f(t)$  in terms of  $b$ .  
 b. If  $\lim_{t \rightarrow 4} f(t)$  exists, determine the value of  $b$ .

## Answers:

1. a)  $\lim_{x \rightarrow -3^-} f(x) = -2$     b)  $\lim_{x \rightarrow -3^+} f(x) = -2$     c)  $\lim_{x \rightarrow -3} f(x) = -2$   
 d)  $\lim_{x \rightarrow 1^+} f(x) = 1$     e)  $\lim_{x \rightarrow 1^-} f(x) = 1$     f)  $\lim_{x \rightarrow 1} f(x) = 1$   
 g)  $\lim_{x \rightarrow 3^-} f(x) = -2$     h)  $\lim_{x \rightarrow 3^+} f(x) = 1$     i)  $\lim_{x \rightarrow 3} f(x)$  does not exist  
 j)  $f(-3)$  is undefined    k)  $f(1) = 1$     l)  $f(3) = -2$

2. a)

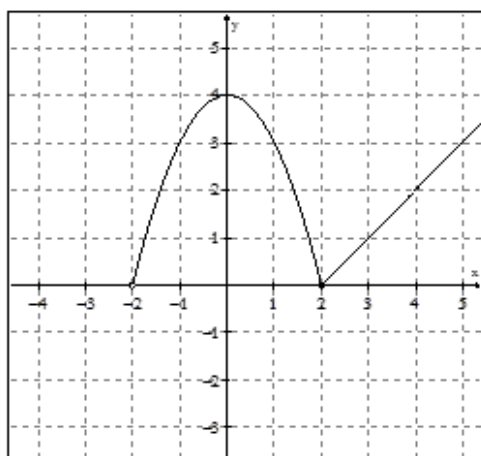


$$\lim_{x \rightarrow 1^-} f(x) = 2 \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = 2$$

Since the limits from the left and right of 1 are equal,

$$\lim_{x \rightarrow 1} f(x) = 2$$

b)

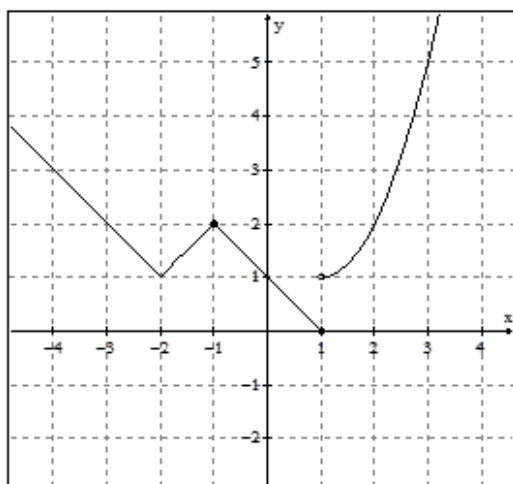


$$\lim_{x \rightarrow 2^-} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 2^+} f(x) = 0$$

Since the limits from the left and right of 2 are equal,

$$\lim_{x \rightarrow 2} f(x) = 0$$

c)



$$\lim_{x \rightarrow 1^-} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = 1$$

Since the limits from the left and right of 1 are not equal,

$$\lim_{x \rightarrow 1} f(x) \text{ does not exist.}$$

3. a)  $\lim_{x \rightarrow -2^+} f(x) = 4$       b)  $\lim_{x \rightarrow 1^-} f(x) = 0$       c) *Since*  $\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^+} f(x) = 4$ ,  
*then*  $\lim_{x \rightarrow -1} f(x) = 4$

$$\lim_{x \rightarrow 0^-} f(x) = 3$$

$$\lim_{x \rightarrow 2^-} f(x) = 0$$

$$\lim_{x \rightarrow 0^+} f(x) = 2$$

$$\lim_{x \rightarrow 2^+} f(x) = 0$$

d)

*Since*  $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$ ,  
*then*  $\lim_{x \rightarrow 0} f(x) \text{ DNE}$

*Since*  $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = 0$ ,  
*then*  $\lim_{x \rightarrow 2} f(x) = 0$

$$\lim_{x \rightarrow 2^-} f(x) = 0$$

$$\lim_{x \rightarrow 2^+} f(x) = 0$$

e)

*Since*  $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = 0$ ,  
*then*  $\lim_{x \rightarrow 2} f(x) = 0$

$$4. \quad f(x) = \begin{cases} \frac{3(-x+2)}{-x+2}, & x \leq 2 \\ \frac{3(x-2)}{-x+2}, & x > 2 \end{cases} = \begin{cases} 3, & x \leq 2 \\ -3, & x > 2 \end{cases}$$

$$a) \lim_{x \rightarrow 2^-} f(x) = 3 \quad b) \lim_{x \rightarrow 2^+} f(x) = -3 \quad c) \text{ Since } \lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x), \text{ then } \lim_{x \rightarrow 2} f(x) \text{ DNE}$$

$$5. \quad a) \text{ A possible defining equation is } f(x) = \begin{cases} 2|x+1| - 2 & x < 0 \\ 3 & 0 \leq x < 2 \\ -\sqrt{x-2} + 3 & x \geq 2 \end{cases}$$

b) i) The limit as  $x$  approaches  $0$  does not exist since the limit from the left does not equal the limit from the right.

$$\lim_{x \rightarrow 0^-} f(x) = 0 \quad \lim_{x \rightarrow 0^+} f(x) = 3 \quad \text{Therefore } \lim_{x \rightarrow 0} f(x) \text{ does not exist}$$

ii) The limit as  $x$  approaches  $2$  equals  $3$  since the limit from the left equals the limit from the right.

$$\lim_{x \rightarrow 2^-} f(x) = 3 \quad \lim_{x \rightarrow 2^+} f(x) = 3 \quad \text{Therefore } \lim_{x \rightarrow 2} f(x) = 3$$

$$6a. \quad \lim_{t \rightarrow 4^+} f(t) = 2 - 16b \quad \text{and} \quad \lim_{t \rightarrow 4^-} f(t) = 12 + b$$

$$b. \quad \text{If } \lim_{x \rightarrow 4} f(t) \text{ exists, then } \lim_{t \rightarrow 4^+} f(t) = \lim_{t \rightarrow 4^-} f(t), \text{ so } 2 - 16b = 12 + b \rightarrow b = -\frac{10}{17}$$