The Mathematical Constant e

The number *e* is of eminent importance in mathematics, alongside 0, 1, π and *i*. All five of these numbers play important and recurring roles across mathematics. Like π, *e* is an irrational number. The value e can be defined a number of ways. We will use the following definition:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

n	e
1	
5	
10	
100	
1000	
etc.	

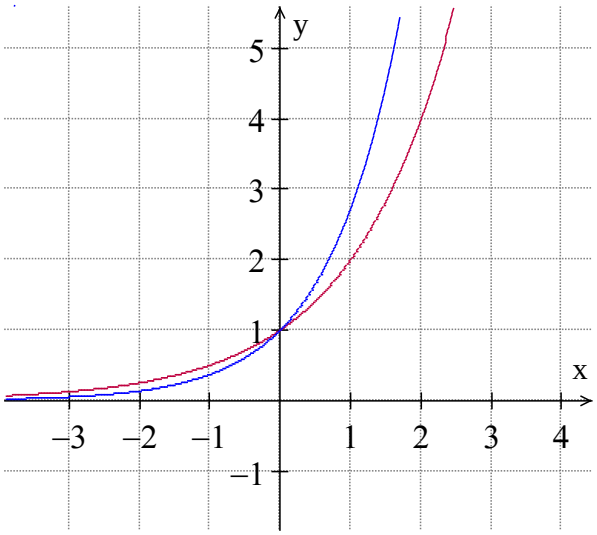
By substituting increasing values of n, we can see that the value of *e* appears to be approximately equal to _____.

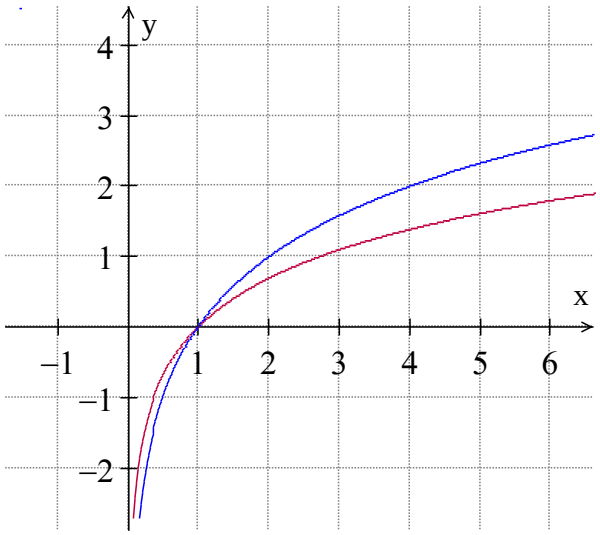
Using a calculator, we can see that the value of *e* evaluated to 5 decimal places is _____.

- Logarithms with base *e* are referred to as "natural logarithms" and we write $\ln x$ which means $\log_e x$.
- Note that the laws of logarithms also apply to natural logarithms.

Example 1: Compare the Graphs of $y = e^x$ to $y = 2^x$ and $y = \ln x$ to $y = \log_2 x$

Label each function below with the appropriate equation.





Example 2: Identify Characteristics of Graphs of Natural Logarithmic Functions

Identify the following characteristics of the graph of each function:

i. the equation of the asymptote
ii. the domain and range
iii. the y-intercept (to one decimal place)
iv. the x-intercept (to one decimal place)

a. $y = \ln(x-5)-4$
b. $y = \ln(-(x-3))+1$

Solution:

a. $y = \ln(x-5)-4$

i) Visualize the transformations of the graph of $y = \ln x$:

The graph of $y = \ln x$ has been _____.

Therefore the equation of the _____ asymptote is _____.

ii) Domain: _____ Range: _____

iii) y-intercept: _____

iv) x-intercept: _____

b. $y = \ln(-(x-3))+1$

i) Visualize the transformations of the graph of $y = \ln x$:

The graph of $y = \ln x$ has been _____.

Therefore the equation of the _____ asymptote is _____.

ii) Domain: _____ Range: _____

iii) y-intercept: _____

iv) x-intercept: _____

Example 3: Evaluate Expressions Containing Natural Logarithms

Evaluate each expression.

a. $4\ln e + 5\ln 1 - \ln e^3$ b. $e^{\ln 12 - 3\ln 2}$

Solution:

a. $4\ln e + 5\ln 1 - \ln e^3$

b. $e^{\ln 12 - 3\ln 2}$

Example 4: Solve Natural Logarithmic Equations

Solve the following equations.

a. $\ln x = 2\ln 4 + \ln 3$ b. $3\ln 2x + 4 = 10$ c. $6 + 5e^{2x} = 21$ d. $\log_2 e^{-4x} = 5$

Solution:

$\ln x = 2\ln 4 + \ln 3$	$3\ln 2x + 4 = 10$	$6 + 5e^{2x} = 21$	$\log_2 e^{-4x} = 5$

Example 5: Solve a Problem Using Natural Logarithms

The temperature, T, in degrees Celsius, of a cup of hot chocolate *t* minutes after it is made is given by the equation $T(t) = 92e^{-0.06t}$.

a. Calculate the temperature of the hot chocolate 8 minutes after it is poured.

b. How long will it take the hot chocolate to cool to 50°C?

Solution:

a. Substitute t = 8 into the equation and solve for T.

b. Substitute T = 50 into the equation and solve for t.

Worksheet - The Mathematical Constant e

1. **Q** Evaluate the following:

a) $\ln e^5$ b) $\ln e^{-4}$ c) $\ln e^{10}$ d) $\ln e^{-1}$

e) $\ln e$ f) $4 \ln e$ g) $2\ ln1$ h) $3 \ln e^2$

2. **Q** Identify the following characteristics of the graph of each function:

i) the domain and range.
ii) The y-intercept (to one decimal place)
iii) The equation of the asymptote
iv) The x-intercept (to one decimal place)

$y = \ln(x - 4) - 2$

$y = -\ln(x + 2) + 3$

3. **Q** Expand $2 \ln\left(\frac{a}{b}\right)$

4. **Q** Simplify each of the following: $\ln 20 - \ln 10$

$2 \ln e - 3 \ln 1 + \ln e^6$

5. **Q** . Evaluate.

a) $e^{\ln 5}$ b) $e^{\ln 10}$ c) $e^{2 \ln 3}$ d) $e^{3 \ln 2}$

e) $e^{-2 \ln 4}$ f) $e^{\ln 10 - \ln 2}$ g) $e^{5 \ln 2}$ h) $e^{-\ln 3}$

i) $e^{\ln 8 + \ln 7}$ j) $e^{4 \ln 5 - 5 \ln 4}$ k) $e^{2 \ln 3 + \ln 5}$

6. **Q** i) Solve for x without a calculator.
ii) Find the approximate solution for x with a calculator.

a) $e^x = 16$ b) $e^{3x} = 27$ c) $e^{2x} = 5$ d) $e^{5x} = 9$

e) $4e^{2x} = 5$ f) $e^{5-3x} = 2$ g) $e^{x+1} = 7$ h) $e^{2x-1} = 3$

i) $4 - 2e^x = -22$

7. **Q** i) Solve for x without a calculator.
ii) Find the approximate solution for x with a calculator.

a) $\ln x = -1$ b) $\ln x = \frac{1}{3}$ c) $\ln x = 5$

d) $\ln(2x - 1) = 3$ e) $\ln(x + 2) = 4$ f) $\ln\left(\frac{1}{x}\right) = 2$

g) $\ln x = \ln 5 + \ln 8$ h) $5 + 2\ln x = 6$ i) $\ln x^2 = 2\ln 4 - 4\ln 2$

j) $-5 + 2\ln 3x = 5$

8. **Q** Solve for x.

a) $e^{\ln(x+1)} = 5$ b) $e^{\ln 5x} = 10$ c) $e^{\ln(4x+2)} = 14$

d) $e^{\ln x^2 + \ln x} = 8$ e) $e^{\ln 2x} = 6$ f) $e^{(2\ln 3)-2} = x$

g) $\log_{10} e^x = 1$ h) $\log_5 e^{2x} = 3$ i) $\ln(e^{2x-1}) = 5$

j) $\ln(e^{5x+2}) = 22$

9. **Q** The growth of a culture of bacteria can be modeled by the equation, $N(t) = N_0 e^{0.105t}$, where $N(t)$ is the number of bacteria after t hours and N_0 is the initial number of bacteria.

a) If the culture has 300 bacteria initially, what is the estimated population in 12 hours?
b) How much time will be required for it to double in population?

10. **Q** A radioactive element decays exponentially according to the formula $A = A_0 e^{-0.04463t}$, where A is the amount present after t days and A_0 is the initial amount. If the initial amount is $80g$,

a) Find the amount remaining after 45 days.
b) After how much time will the radioactive element decay to 20% of the initial amount?

Worksheet answers:

1 a. 5 b. -4 e. 1 g. 0 h. 6

2 a. i) Domain: $x > 4$, $x \in R$, Range: $y \in R$ b. i) Domain: $x > -2$, $x \in R$, Range: $y \in R$

ii) no y-intercept ii) y-intercept = 2.31

iii) $x = 4$ iii) $x = -2$

iv) x-intercept = 11.4 iv) x-intercept = 18.1

3. $2\ln a - 2\ln b$

4. a. $\ln 2$ b. 8

5 b. 10 d. 8 e. 1/16 h. 1/3 j. 0.61 k. 45

6 a. $x = 2.8$ c. $x = 0.8$ e. $x = 0.11$ f. $x = 1.4$ i. $x = 2.56$

7 b. $x = 1.4$ d. $x = 10.54$ f. $x = 0.135$ g. $x = 40$ i. $x = 1, -1$ j. $x = 49.47$

8 c. $x = 3$ d. $x = 2$ f. $x = 1.22$ h. $x = 2.41$ i. $x = 3$

9 a. 1058 bacteria b. 6.6 hours

10 a. 10.7 g b. 36 days