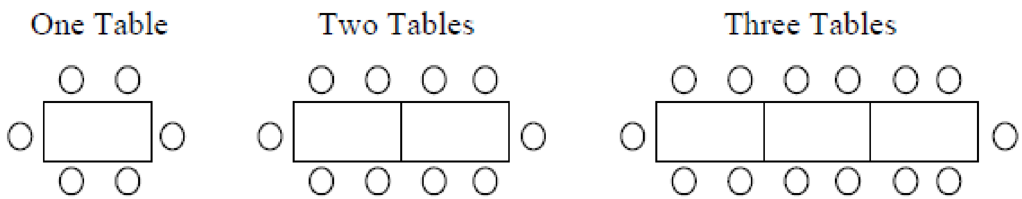
Arithmetic Sequences

Hannah is organizing the seating in a long narrow banquet hall. She will be placing tables end to end and she wants to determine the number of people who can be seated using this strategy.



If she uses 1 table, she can seat 6 people. If she uses 2 tables, she can seat 10 people. With 3 tables, she can seat 14 people.

If we list these numbers, and continue the pattern, we get _____, _____, _____, _____, _____, _____, …

An ordered list of numbers, such as 6, 10, 14, 18, 22, 26, …, that follow a pattern is called a **sequence**.
Each item in the sequence is a **term**. Each term in a sequence has a position or **term number** and a **term value**.
For the sequence 6, 10, 14, 18, 22, 26, …

$t_1 = 6$ The first term, t_1, is equal to 6. The term number is 1 and the term value is 6.
$t_2 = 10$ The second term, t_2, is equal to 10. The term number is 2 and the term value is 10.
$t_3 = 14$ The third term, t_3, is equal to 14. The term number is 3 and the term value is 14.
etc…

Term numbers are usually restricted to the natural numbers (i.e. 1, 2, 3, …) because they represent the position of the term in the sequence.

The **number of terms** in the sequence is n.

The sequence $t_1, t_2, t_3, \ldots, t_n$ is known as a **finite sequence** since it has a *finite* number of terms. For example: {10, 20, 30, 40, 50, …, 90} is a finite sequence.

The sequence $t_1, t_2, t_3, \ldots, t_n, t_{n+1}, \ldots$ is known as an **infinite sequence** since it has an *infinite* number of terms. For example: {15, 10, 5, 0, –5, …} is an infinite sequence.

Simply listing the first few terms of a sequence is not sufficient to define a unique sequence; the n^{th} term must be given to make the sequence truly unique. Consider the following example:

Given the sequence 2, 4, 6, …, what are the next three terms?

- Mon says that the sequence should read 2, 4, 6, 8, 10, 12, … stating that each term is increasing by 2.

- Meaghan says that the sequence should read 2, 4, 6, 10, 16, 26, … . She states that you add the two preceding terms to get the next term (i.e. $t_1 + t_2 = t_3,\ t_2 + t_3 = t_{4, \ldots}$). This is also an acceptable answer.

To avoid this problem, we can write an expression for the n^{th} term. This is known as the **general term** and is represented by t_n.

- If we wanted to generate Mon's sequence, we could use the *explicit* formula $t_n = 2n,\ n \in N$.

- If we wanted to generate Meaghan's sequence, we could use the *recursive* formula $t_n = t_{n-2} + t_{n-1};\ n \in N,\ n \geq 3,\ t_1 = 2,\ t_2 = 4$.

The **general term** is an expression that can be used to calculate the value of any term in a sequence.

ARITHMETIC (LINEAR) SEQUENCES

An arithmetic, or linear, sequence is a sequence of numbers in which each term can be found by adding or subtracting a constant amount to the preceding term. This constant amount is known as the **common difference**, d, between successive terms in an arithmetic sequence. To calculate d, use $d = t_n - t_{n-1}$.

For example, in the sequence 6, 10, 14, 18, …, each term is _______________ than the previous term. The common difference is ______.

Determining the General Term for an Arithmetic Sequence

Consider the arithmetic sequence 6, 10, 14, 18, … from above.

Using the *first term* and the *common difference*, we can write the term values of this sequence as follows:

$t_1 = 6 + 4 \times 0$
$t_2 = 6 + 4 \times 1$
$t_3 = 6 + 4 \times 2$
$t_4 = 6 + 4 \times 3$
…
$t_n =$ _______________ $=$ _______________

FORMULA FOR THE EXPLICIT *GENERAL TERM* OF ANY *ARITHMETIC* SEQUENCE

$t_n =$ ____________________ $t_n =$ general term t_1 = first term $d =$ common difference $n =$ term number

By substituting the appropriate value of n, we can use the general term to calculate the value of *any* term in the sequence. For example, in the above sequence, to calculate the value of the *tenth* term we would substitute n = 10 into the general term:

$$\begin{aligned} t_n &= 6 + 4(n-1) \\ t_{10} &= 6 + 4(10-1) \\ t_{10} &= 6 + 4(9) \\ t_{10} &= 6 + 36 \\ t_{10} &= 42 \end{aligned}$$

OR

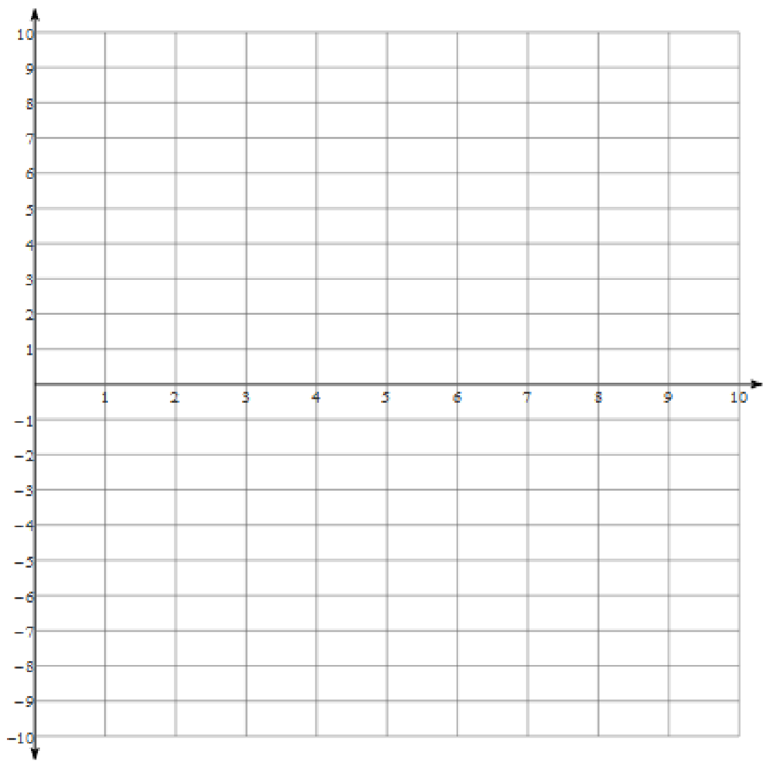
$$\begin{aligned} t_n &= 4n + 2 \\ t_{10} &= 4(10) + 2 \\ t_{10} &= 40 + 2 \\ t_{10} &= 42 \end{aligned}$$

Example 1: Investigating the Graphs of Arithmetic Sequences

Term Number (n)	1	2	3	4	5	6
Term Value (t_n)	7	4	1	–2	–5	–8

a. Sketch a graph of the term values, t_n, versus the term numbers, n.
b. What type of relationship is exhibited in the graph?
c. Determine the equation of this function.
d. Should the graph be completed using discrete or continuous data? Justify your response.
e. Determine the explicit formula for this arithmetic sequence.
f. Compare the equation of the linear function to the explicit formula that is used to describe the arithmetic sequence given above.

Solution:

a. 

b. The type of relationship exhibited in the graph above is ___________________________.

c. Equation of this function: ____

d. The graph shows _____________________ data since ____.

e. Explicit formula for this arithmetic sequence: ____

f. The equation of the linear function is equivalent to the explicit formula that is used to describe the arithmetic sequence. Note that the slope of the linear function represents the _________________________ of the arithmetic sequence.

Example 2: Determine a Particular Term

A visual and performing arts group wants to hire a community events leader. The person will be paid \$12 for the first hour of work, \$19 for two hours of work, \$26 for three hours of work, and so on.

a. Write the general term that you could use to determine the pay for any number of hours worked.
b. What will the person get paid for 6 hours of work?

Solution:

a. Sequence: ______, ______, ______, …

$t_1 =$ _______ d = _______

$t_n = t_1 + d(n-1)$

The general term of the sequence is __________________

b. To determine the pay for 6 hours of work, we must find the _________ term in the sequence.

Example 3: Determine the Number of Terms in an Arithmetic Sequence

The musk-ox and the caribou of northern Canada are hoofed mammals that survived the Pleistocene Era, which ended 10 000 years ago. In 1955, the Banks Island musk-ox population was approximately 9250. Suppose that in subsequent years, the growth of the musk-ox population generated an arithmetic sequence, in which the number of musk-ox increased by approximately 1650 each year. In which year did the musk-ox population reach 100 000?

Solution:

Sequence: ____________, ____________, ____________, …

$t_1 =$ _____________ d = _____________

$t_n = t_1 + d(n-1)$

It would take _______ years for the musk-ox population to reach 100 000, in the year __________.

Example 4: Determine t_1, t_n, and n

Tom has a part-time job at the local grocery store. He has been asked to create a display of cereal boxes. Tom decides to build a pyramid with the cereal boxes. The numbers of boxes in the rows produce an arithmetic sequence. If there are 16 boxes in the third row from the bottom and 6 boxes in the eighth row from the bottom, algebraically solve the following:

a. How many boxes are in the bottom row?
b. Determine the general term, t_n, for the sequence.
c. Determine the total number of rows in Tom's display if the top row has two boxes.

Solution:

a. How many boxes are in the bottom row?

b. Determine the general term, t_n, for the sequence.

c. Determine the total number of rows in Tom's display if the top row has two boxes.