Geometric Series

Geometric Series – The sum of the terms of a geometric sequence

Paul sees a new band at a concert. He e-mails a link for the band's Web site to five of his friends. They each forward the link to five of their friends. This pattern continues with each person e-mailing five other people.

At every level, the total number of people receiving the link is five times the number of people receiving the link in the previous level. The pattern can be modeled by a geometric series where the first term is 5 and the common ratio is 5. The series for this situation would be _______ + _______ + _______ + _______, which gives a sum of __________ people contacted by e-mail after 4 rounds of e-mails.

If we were to extend this series to 20 or 100 rounds of e-mails, a more efficient method (other than just adding the terms) for determining the sum of the series is needed.

To develop a formula for the sum of a geometric series, consider the following:

- List the above original series:
  $S_4 = 5 + 25 + 125 + 625 \rightarrow$ Equation 1

- Multiply equation 1 by the common ratio, 5:
  $5(S_4 = 5 + 25 + 125 + 625)$
  $5S_4 = 25 + 125 + 625 + 3125 \rightarrow$ Equation 2

- Subtract Equation 1 from Equation 2 and then solve for S_4:

$$\begin{array}{rl} 5S_4 = & \quad\;\; 25 + 125 + 625 + 3125 \\ -S_4 = & 5 + 25 + 125 + 625 \\ \hline (5-1)S_4 = & -5 + 0 + 0 + 0 + 3125 \\ 4S_4 = & 3120 \\ S_4 = & 780 \end{array}$$

In general, a geometric series may be expressed as:

$S_n = t_1 + t_1r + t_1r^2 + t_1r^3 + \ldots + t_1r^{n-1} \rightarrow$ *Equation* 1

Multiply equation 1 by the common ratio, r:
$rS_n = \quad t_1r + t_1r^2 + t_1r^3 + t_1r^4 + \ldots + t_1r^n \rightarrow$ *Equation* 2

Subtract Equation 1 from Equation 2 and solve for S_n:

$$\begin{array}{rl} rS_n = & \qquad t_1r + t_1r^2 + t_1r^3 + t_1r^4 + \ldots + t_1r^n \\ -S_n = & t_1 + t_1r + t_1r^2 + t_1r^3 + \ldots + t_1r^{n-1} \\ \hline (r-1)S_n = & -t_1 + 0 + 0 + 0 + 0 + \ldots + 0 + t_1r^n \end{array}$$

$$S_n = \frac{-t_1 + t_1r^n}{r-1} = \frac{t_1r^n - t_1}{r-1} = \frac{t_1\left(r^n - 1\right)}{r-1}$$

FORMULAS FOR THE SUM OF A GEOMETRIC SERIES

$S_n = \frac{t_1(r^n - 1)}{r - 1},\ r \neq 1$	OR $S_n = \frac{rt_n - t_1}{r - 1},\ r \neq 1$
S_n = sum of the first n terms	S_n = sum of the first n terms
t_1 = first term	t_1 = first term
r = common ratio	r = common ratio
n = number of terms	t_n = n^{th} term

Example 1: Find the First Term of a Geometric Series

Find t_1 in the geometric series for which $S_7 = 13116$ and r = 3.

Solution:

Example 2: Determine the Sum of a Geometric Series

Determine the sum of each geometric series.

a. $t_1 = 5,\ r = 2,\ n = 8$ b. $12 - 6 + 3 - 1.5 + ... + t_{11}$

Solution:

a. $t_1 = 5,\ r = 2,\ n = 8$ b. $12 - 6 + 3 - 1.5 + ... + t_{11}$

Example 3: Determine the Number of Terms in a Geometric Series

If $\frac{5}{4}+\frac{5}{6}+\frac{5}{9}+\ldots+t_n \doteq 3.7066$, calculate the number of terms in the series.

Solution:

Example 4: Find the Sum of a Geometric Series for an Unspecified Number of Terms

Determine the sum of each geometric series.

a. $\frac{1}{64}+\frac{1}{16}+\frac{1}{4}+\ldots+1024$

b. $-2+4-8+\ldots-8192$

Solution:

a. $\frac{1}{64}+\frac{1}{16}+\frac{1}{4}+\ldots+1024$

b. $-2+4-8+\ldots-8192$

Example 5: Apply Geometric Series

A tennis tournament has 256 players. When players win their match, they go on to play another match. If they lose their match, they are out of the tournament. What is the total number of matches that will be played in this tournament?

Example 6: Geometric Sum in Sigma Notation

Evaluate $\sum_{k=3}^{10} 4(2)^{k-1}$.

Solution:

Method 1	Method 2
Find each term in the indicated series and then add the terms. $\sum_{k=3}^{10} 4(2)^{k-1}$	Use a geometric series formula. $\sum_{k=3}^{10} 4(2)^{k-1}$

Extra Practice

1. Find the sum of each geometric series.

   a. $\sum_{k=1}^{6} 3(4)^{k-1}$

   b. $\sum_{n=1}^{8} 4\left(\frac{1}{2}\right)^{n-1}$

2. Express each of the following geometric series in sigma notation and then find the sum.

   a. $3+6+12+24+\ldots+49152$

   b. $5+1+\frac{1}{5}+\ldots+\frac{1}{78125}$