Inverse of a Relation

An inverse is often thought of as "undoing" or "reversing" a position, order, or effect. Whenever you undo something that you or someone else did, you are using an inverse (for example, unwrapping a gift that someone else wrapped, closing a door that has just been opened, etc.).

In mathematics, the inverse of a function reverses the process represented by that function. For example, the process of squaring a number is reversed by taking the square root. The process of taking the reciprocal of a number is reversed by taking the reciprocal again.

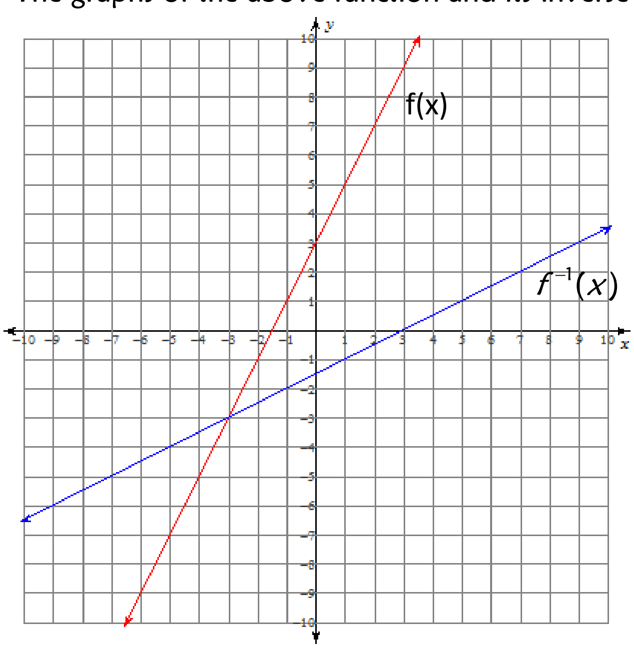
Consider the function $f(x)=2x+3$. With this function, if you substitute a value in for x you would first multiply that value by 2 and then add 3. To create the inverse, you would first subtract 3 and then divide by 2.

Function: $f(x)=2x+3$			**Inverse:** $f^{-1}(x)=\frac{(x-3)}{2}$	
x	y		x	y
0	3		3	
1	5		5	
2	7		7	
3	9		9	
4	11		11	
5	13		13	

Notice that the x- and y-coordinates of the ordered pairs of the inverse can be found by interchanging the x- and y-coordinates of the ordered pairs of the original function.

Mapping rule: $(x,y)\rightarrow(y,x)$

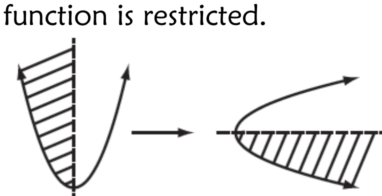
The graphs of the above function and its inverse are shown below:



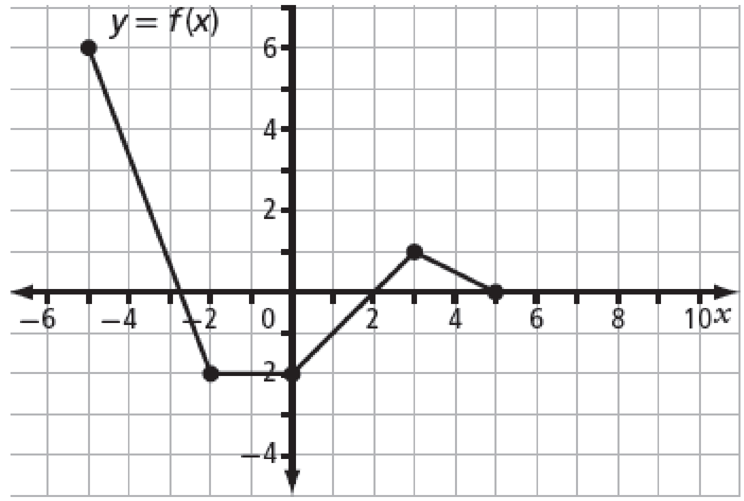
The graphs of a relation and its inverse are reflections of each other in the line y = x.

When working with an equation of a function $y=f(x)$, interchange x for y, then solve for y to get an equation for the inverse. If the inverse is a function, then we use the notation $y=f^{-1}(x)$.

However, the inverse of a function is not necessarily a function. If this is the case, we can restrict the domain of the base function so that the inverse becomes a function. We will see this frequently with any function that changes direction (increasing to decreasing, or vice versa) at some point in the domain of the function. For example, the inverse of $f(x)=x^2, x\geq 0$ is $f^{-1}(x)=\sqrt{x}$. The inverse will be a function only if the domain of the base function is restricted.



Example 1: Graph an Inverse



Consider the graph of the relation shown.

a. Sketch the graph of the inverse relation.

b. State the domain and range of the relation and its inverse.

c. Determine whether the relation and its inverse are functions.

Solution:

a. To graph the inverse relation, interchange the x-coordinates and y-coordinates of key points on the graph of the relation.

Points on the Relation	Points on the Inverse Relation
(–5, 6)	
(–2, –2)	
(0, –2)	
(3, 1)	
(5, 0)	

Sketch the graph of the inverse.

The graphs are reflections of each other in the line $y = x$. Sketch this line.

Are there any invariant points? ____________________

b. The domain of the relation becomes the range of the inverse relation and the range of the relation becomes the domain of the inverse relation.

	Domain	Range
Relation		
Inverse Relation		

c. The original relation is a function since for every value of x in the domain there is only one value of y in the range. In other words, the graph of the relation passes the vertical line test.

The inverse relation __________ a function because it __________ the vertical line test. There is more than one value of y in the range for at least one value of x in the domain. To confirm this you can use the horizontal line test on the graph of the original relation.

Horizontal Line Test: *A test used to determine if the graph of an inverse relation will be a function. If it is possible for a horizontal line to intersect the graph of a relation more than once, then the inverse of the relation is not a function.*

Example 2: Restrict the Domain

Consider the function $f(x) = x^2 + 3$.

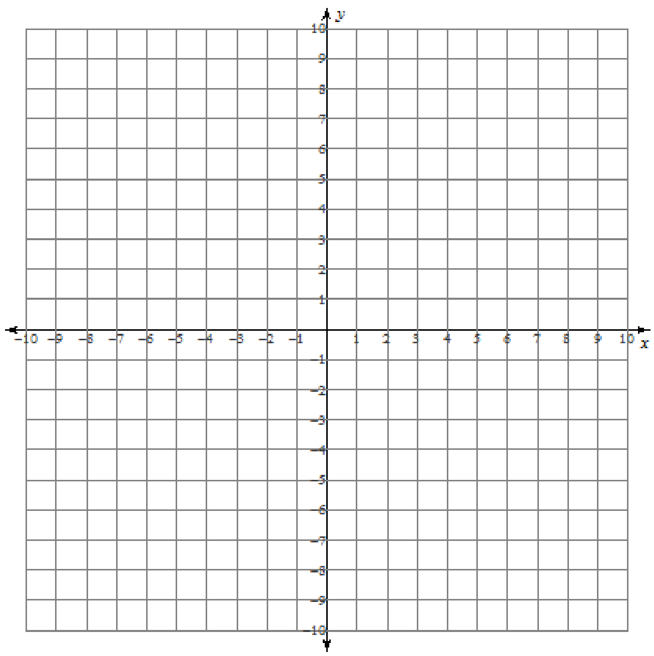
a. Graph the function f(x). Is the inverse of f(x) a function?

b. Graph the inverse of f(x) on the same set of coordinate axes.

c. Describe how the domain of f(x) could be restricted so that the inverse of f(x) is a function.

Solution:

a. Sketch the graph of $f(x) = x^2 + 3$.

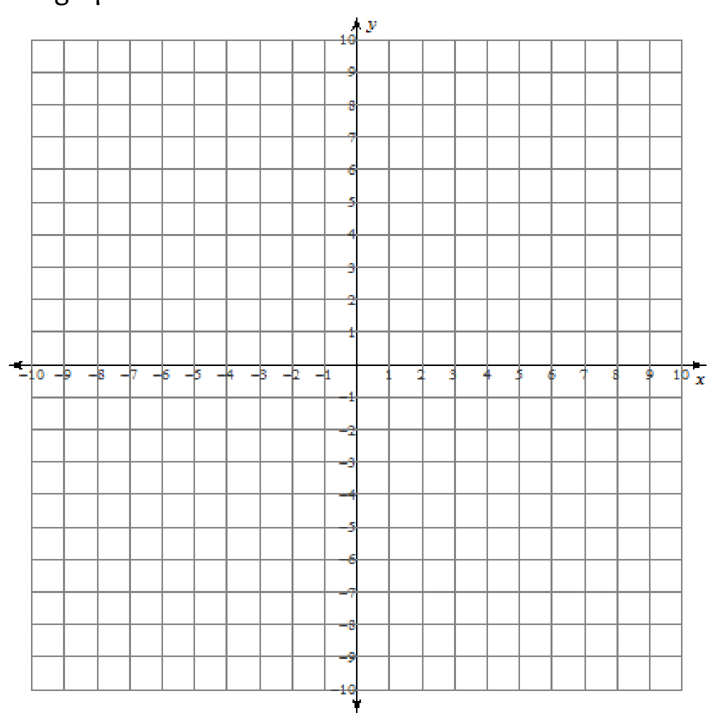


Since the graph of the function ____________ the horizontal line test, the *inverse* of $f(x)$ ____________ a function.

b. Use key points on the graph of $f(x)$ to help you sketch the graph of the inverse.

c. The inverse of $f(x)$ is a function if the graph of $f(x)$ passes the horizontal line test. One possibility is to restrict the domain of $f(x)$ so that the graph is only one half of the parabola. Since the equation of the axis of symmetry is x = 0, restrict the domain to ________________.

The inverse of $f(x) = x^2 + 3$, $x \geq 0$ is a function.



Example 3: Determine the Equation of the Inverse

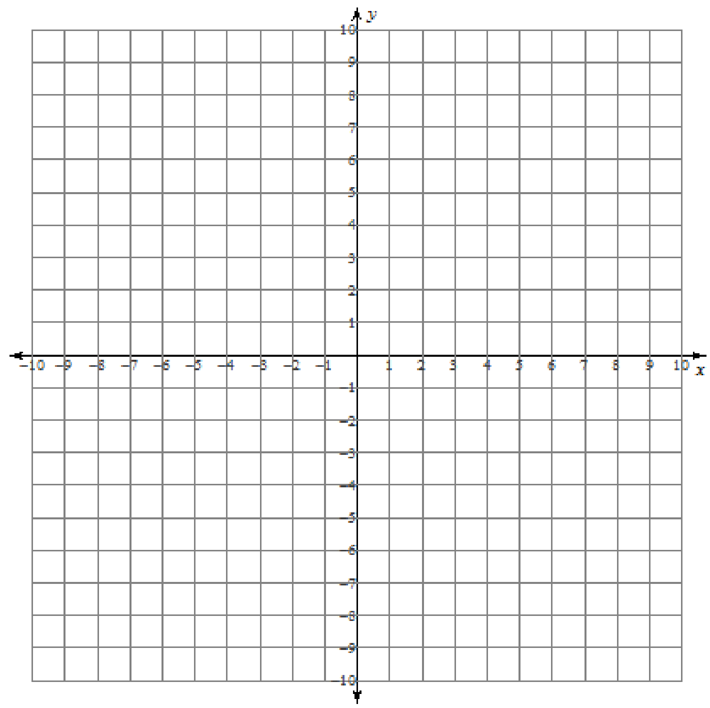
Algebraically determine the equation of the inverse of each function. Verify graphically that the relations are inverses of each other.

a. $f(x) = 3x - 1$ b. $f(x) = (x+3)^2 - 1$

Solution:

a. $f(x) = 3x - 1$

Let $y = f(x)$. To find the equation of the inverse, interchange x and y, and then solve for y.

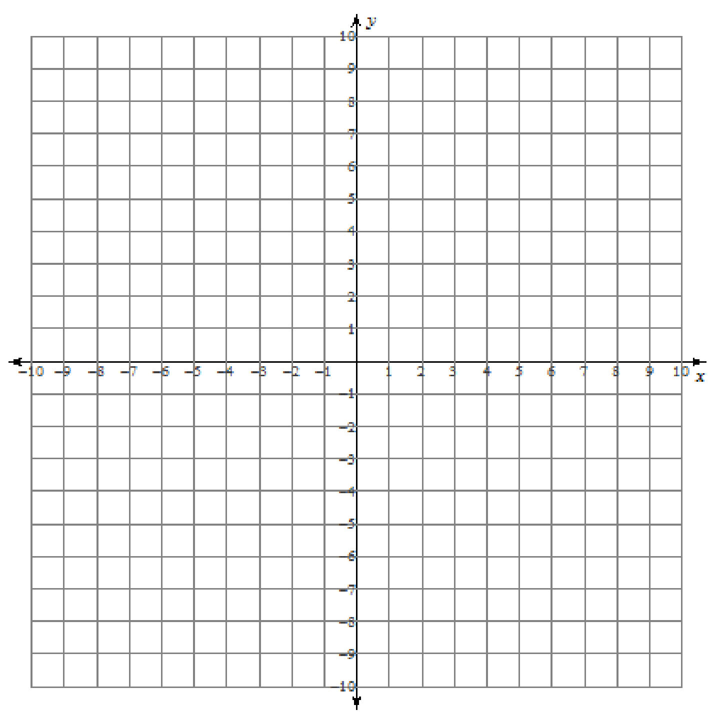


Graph the original function and its inverse on the same set of axes.

Since the graphs are reflections of each other in the line y = x, the functions are inverses of each other.

b. $f(x) = (x+3)^2 - 1$

Let $y = f(x)$. To find the equation of the inverse, interchange x and y, and then solve for y.



Graph the original function and its inverse on the same set of axes.

Since the graphs are reflections of each other in the line y = x, the relations are inverses of each other, however, $y = -3 \pm \sqrt{x+1}$ is not a function.

How could you restrict the domain of the function $f(x) = (x+3)^2 - 1$ so the inverse would be a function? ______________________________