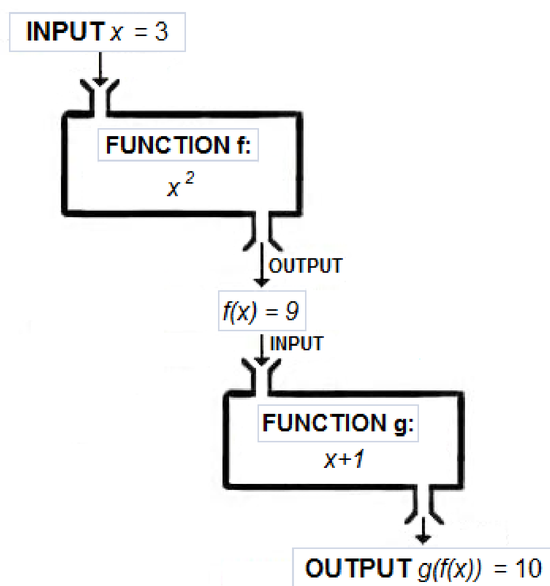
Composite Functions

- *Composite functions are functions that are formed from two or more functions, say, f(x) and g(x), in which the output of one function is used as the input for the other function.*
- *The composition of f(x) and g(x) can be written as f(g(x)) or $(f \circ g)(x)$ and is read as "f of g of x".*
- *Composition of functions must not be confused with multiplication, that is, $(f \circ g)(x)$ does not mean $(f \bullet g)(x)$.*
- *In general, composition of functions will not be commutative, that is f(g(x)) ≠ g(f(x)), except in special cases.*
- *The equation of f(g(x)) is formed when the equation of g(x) is substituted into the equation of f(x).*
- *f(g(x)) exists only for those values of x in the domain of g for which g(x) is in the domain of f.*

To illustrate how composite functions work, consider the following example:

If $\begin{matrix} f(x) = x^2 \\ g(x) = x+1 \end{matrix}$, **then determine g(f(3))**



- First determine f(3):
  $f(x) = x^2$
  $f(3) = (3)^2$
  $f(3) = 9$
- Then determine g(9):
  $g(x) = x+1$
  $g(9) = 9+1$
  $g(9) = 10$

Now determine f(g(3))

- First determine g(3):
  $g(x) = x+1$
  $g(3) = 3+1$
  $g(3) = 4$
- Then determine f(4):
  $f(x) = x^2$
  $f(4) = (4)^2$
  $f(4) = 16$

So g(f(3)) = 10, *but* f(g(3))= 16

Example 1: Evaluate a Composite Function

If f(x)= 4 – x and $g(x) = x^2 + x$, determine each value.

a. f(g(2)) b. g(f(–3)) c. g(g(1))

Solution:

Method 1: Determine the Value of the Inner Function and Then Substitute

a. Evaluate g(2):	b. Evaluate f(–3):	c. Evaluate g(1):
Then substitute this value into f(x):	Then substitute this value into g(x):	Then substitute this value into g(x):
f(g(2)) = __________	g(f(–3)) = __________	g(g(1)) = __________

Method 2: Determine the Composite Function and then Substitute

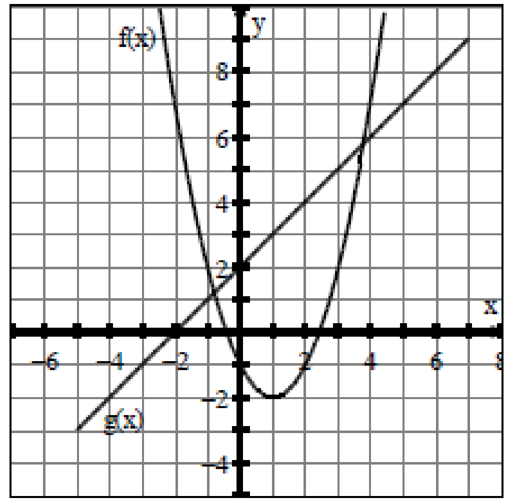
Determine the composite function f(g(x)):	Determine the composite function g(f(x)):	Determine the composite function g(g(x)):
Then substitute x = 2 into f(g(x)):	Then substitute x = –3 into g(f(x)):	Then substitute x = 1 into g(g(x)):
f(g(2)) = __________	g(f(–3)) = __________	g(g(1)) = __________

Example 2: Evaluate a Composite Function by Viewing the Graphs

Use the graphs shown to evaluate each of the following.

a. f(g(1)) c. $(f \circ g)(-1)$

b. g(g(2)) d. $(g \circ f)(3)$



Solution:

a. f(g(1)) = c. $(f \circ g)(-1) =$

b. g(g(2)) = d. $(g \circ f)(3) =$

The Domain and Range of f(g(x))

The composition of two functions exists only where the range of the first function overlaps, or is contained in, the domain of the second function.

The following example illustrates this concept. If functions g(x) and f(x) are defined by the tables on the far left, then the composite function f(g(x)) is defined by the table on the far right.

x	$g(x)$
−1	2.5
0	3
1	−2
2	0
3	1
4	−3

x	$f(x)$
−1	4
0	3
1	2
2	1
3	0
4	−1

x	$g(x)$
−1	2.5
0	3
1	−2
2	0
3	1
4	−3

$g(x)$	$f(g(x))$
2.5	undefined
3	0
−2	undefined
0	3
1	2
−3	undefined

x	$(f \circ g)(x)$
0	0
2	3
3	2

Finding the *domain* of a composite function consists of two steps.

Step 1: Find the domain of the "inside" (input) function. If there are any restrictions on the domain, *keep them.*

Step 2: Construct the composite function. Find the domain of this new function. If there are restrictions on this domain, *add them to the restrictions from Step 1.*

Example 3: Finding the Domain of a Composite Function

Determine the domain of the composite function in each situation.

a. Determine the domain of $f(g(x))$ if $f(x) = x^2 + 2$ and $g(x) = \sqrt{7-x}$.

Step 1: What is the domain of the inside function, g(x)? ________________________________

Step 2: Determine the equation of the composite function, $f(g(x))$

Are there any additional restrictions on the domain of the composite function? _______

Domain of $f(g(x))$: ________________________________

b. Determine the domain of $g(f(x))$ if $f(x) = x^2 + 2$ and $g(x) = \sqrt{7-x}$.

Step 1: What is the domain of the inside function, f(x)? ________________________________

Step 2: Determine the equation of the composite function, $g(f(x))$

New restrictions on the domain: ______________________________

Domain of $g(f(x))$: ________________________________

c. Determine the domain of $f(g(x))$ if $f(x) = \frac{3x}{x-1}$ and $g(x) = \frac{2}{x}$.

Step 1: What is the domain of the inside function, g(x)? ________________________________

Step 2: Determine the equation of the composite function, $f(g(x))$

Additional restrictions on the domain: _____________________

Domain of $f(g(x))$: ________________________________

Example 4: Compose Functions with Restrictions

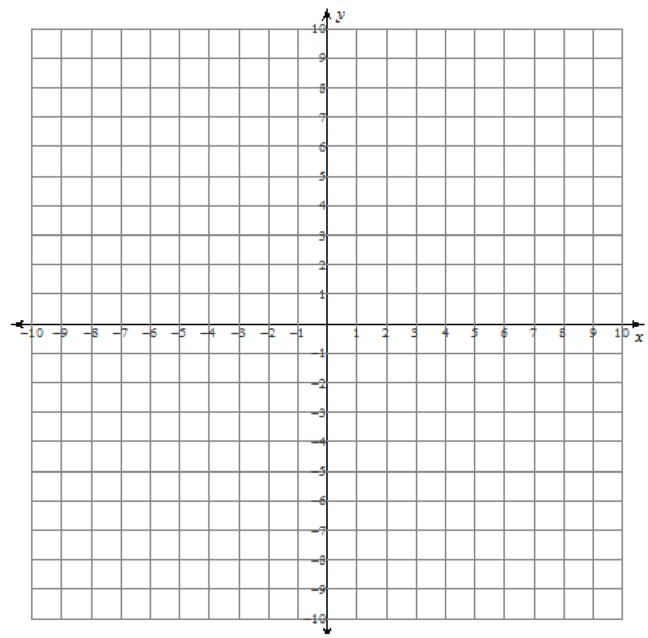
If $f(x) = \sqrt{x-4}$ and $g(x) = x^2$, determine the equation, state the domain and range, and graph each composite function.

a. $(f \circ g)(x)$

b. $(g \circ f)(x)$

Solution:

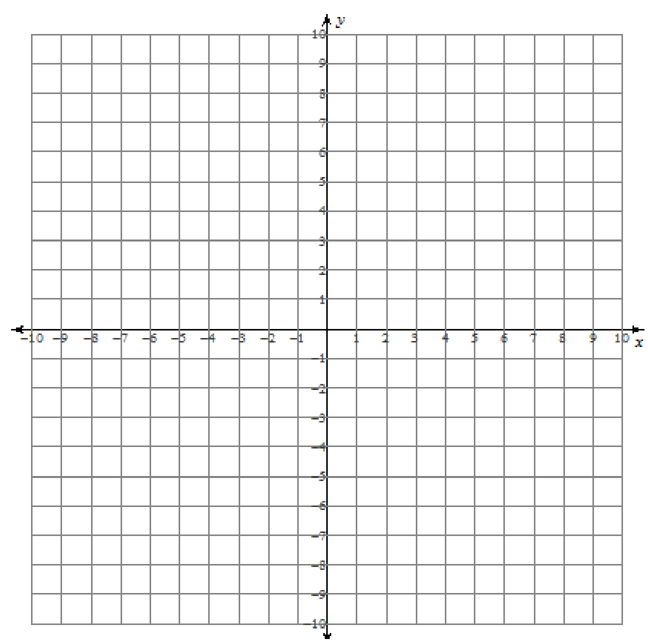
a. Determine $(f \circ g)(x)$



Domain: ___________________________

Range: _____________________________

b. Determine $(g \circ f)(x)$



Domain: ___________________________

Range: _____________________________

Example 5: Determine the Original Functions from a Composition

Determine two functions, f(x) and g(x), where h(x) = f(g(x)).

a. $h(x)=\sqrt{3x-2}$ b. $h(x)=(x-4)^2+3(x-4)+4$ c. $h(x)=x^2-6x+9$ d. $h(x)=5x^2+10x+4$ if $f(x)=5x-1$

Solution:

a. $h(x)=f(g(x))=\sqrt{3x-2}$

Start by letting the "inner function" $g(x)=$_____________. Then, work backward to determine f(x).

The two functions are f(x) = _______________ and g(x) = _______________.

b. $h(x)=f(g(x))=(x-4)^2+3(x-4)+4$

Start by letting $g(x)=$_____________. Then, work backward to determine f(x).

The two functions are f(x) = ______________________and g(x) = ______________________.

c. $h(x)=x^2-6x+9=$ ____________ $=f(g(x))$

Write h(x) in factored form. Let g(x) = __________. Then, work backward to determine f(x).

The two functions are f(x) = _______________ and g(x) = _______________.

d. $h(x)=5x^2+10x+4$ and $f(x)=5x-1$

Let f(g(x))= 5g(x) – 1 = $5x^2+10x+4$. Then, rearrange to determine g(x).

The two functions are f(x) = _______________ and g(x) = _______________.

Example 6: Application of Composite Functions

The temperature as you descend a mine shaft is a function of the depth below the surface. An equation expressing the relationship is $T(d) = 0.01d + 20$, where T is the temperature, in degrees Celsius, and d is the depth, in metres.

a. If you go down a mine shaft at a rate of 4m/s, express the temperature as a function of the time, t, in seconds.

b. What is the temperature after 1 minute of traveling down the mine shaft?

Solution:

a. Since the rate of travel is 4m/s, the distance, in metres, traveled in t seconds is d(t) = ________________.

Since the temperature, T, is a function of depth, d, you can compose the two functions.

T(d) = 0.01d +20

T(d(t)) =

The temperature expressed as a function of time is T(d(t)) = ________________________

b. To determine the temperature after 1 minute of travel, substitute t = __________ in the composite function.

T(d(t)) = 0.04t + 20

The temperature after one minute of travelling down the mine shaft is _________________________.

Example 7: Application of Composite Functions

In the mail you receive a coupon for \$15 off a pair of jeans. When you arrive at the store, you find that all jeans are 25% off.

a. If x is the regular price of a pair of jeans, write a function f(x) that represents the price of the jeans after the coupon is applied.

b. If x is the regular price of a pair of jeans, write a function g(x) that represents the price of the jeans after the 25% discount is applied.

c. Write a composite function that represents the price of the jeans if the 25% discount is applied first and the \$15 coupon is applied second.

d. Write a composite function that represents the price of the jeans if the \$15 coupon is applied first and the 25% discount is applied second.

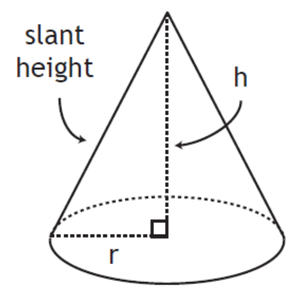
e. You find a pair of jeans that have a regular price of \$65. Use your composite functions from (c) and (d) to determine whether the \$15 coupon or the 25% discount should be applied first in order to result in the lower price.

Extra Practice:

1. Determine the domain of $[f \circ g](x)$ given that $f(x) = \sqrt{x-4}$ and $g(x) = \frac{1}{x^2}$.

2. The surface area and volume of a right cone are:

$$SA = \pi r^2 + \pi rs$$
$$V = \frac{1}{3}\pi r^2 h$$



where r is the radius of the circular base, h is the height of the apex, and s is the slant height of the side of the cone.

Given that a particular cone has a height that is $\sqrt{3}$ times larger than the radius,

a. Express the *apex height* in terms of r.

b. Express the *slant height* in terms of the single variable, r.

c. Write the *volume* formula as a formula in terms of the single variable, r.

d. Write the *surface area* formula as a formula in terms of the single variable, r.

e. Calculate the *volume* of the cone if the radius of its circular base is 6 cm.