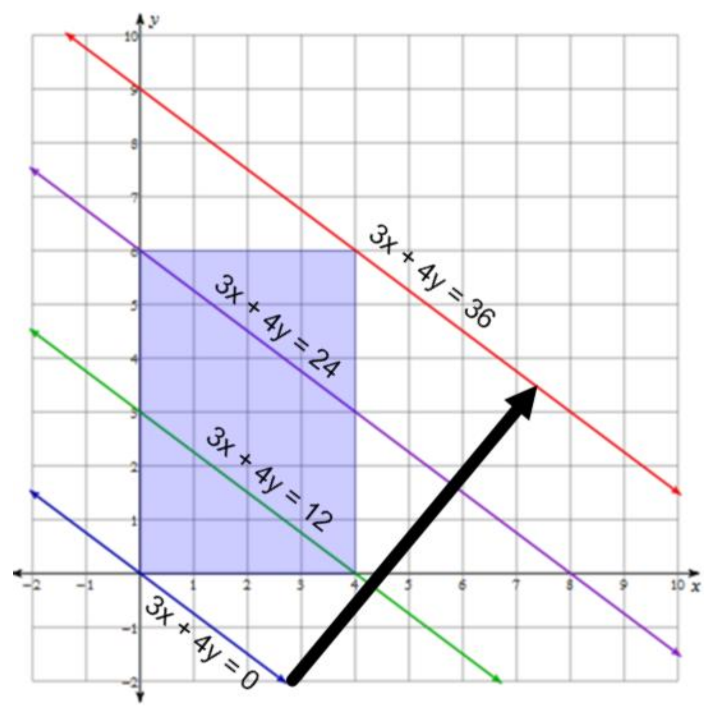
Optimization Problems

Optimization Problem
An optimization problem is a problem where a quantity must be optimized (maximized or minimized) following a set of constraints.

Objective Function
An objective function is an equation for a quantity that is to be optimized. An objective function can be used to solve such problems as:
- maximizing profit
- minimizing cost
- maximizing total quantities
- minimizing resource use

Feasible Region
The graph of the solution of a system of inequalities that form the constraints of a problem is called the feasible region. A feasible region is ***bounded*** if it is completely enclosed by line segments and forms a polygon. If it extends infinitely in any direction, it is **unbounded**. The points where the line segments of the feasible region intersect are the ***vertices*** of the feasible region. You can find the coordinates of a vertex by looking at the graph or by solving the system of the two equations that intersect there.

Maximum and Minimum Values



If the feasible region is bounded, then the objective function has a maximum and a minimum value. These occur at the vertices of the feasible region.

Suppose the feasible region is the rectangle shown in the figure, and the objective is to maximize or minimize the value of C when C = 3x + 4y.

Think of sliding the line from the lower left to the upper right. As the line slides from one edge of the feasible region to the other, the value of 3x + 4y increases.

The *minimum* value of 3x + 4y is 0, which occurs at the lower left vertex.

The *maximum* value of 3x + 4y is 36, which occurs at the upper right vertex.

Example 1: Maximizing an Objective Function

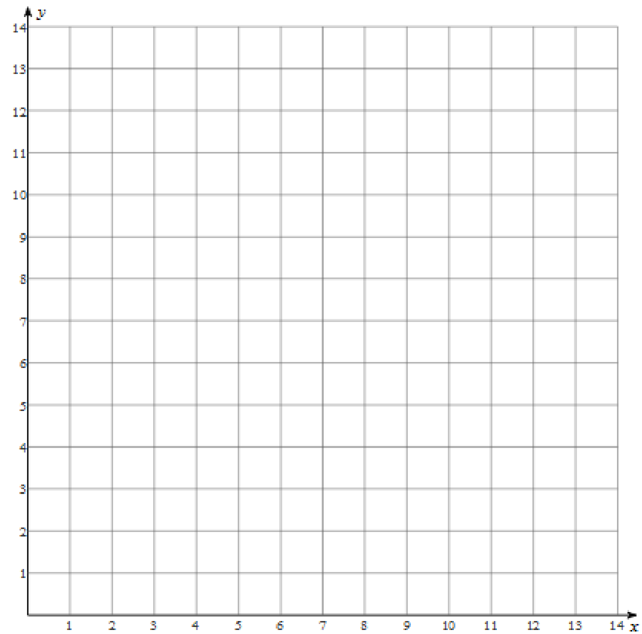
Determine the *maximum* value of P = 3x + 7y for the given constraints.

$x + y \le 10$

$4x + 3y \le 36$

$x \ge 0$

$y \ge 0$



STEP 1: Graph the constraints and shade the feasible region.

STEP 2: Calculate the value of P at each vertex of the feasible region.

(x, y)	P = 3x + 7y

The *maximum* value of P is ______, which occurs at _______.

Example 2: Minimizing an Objective Function

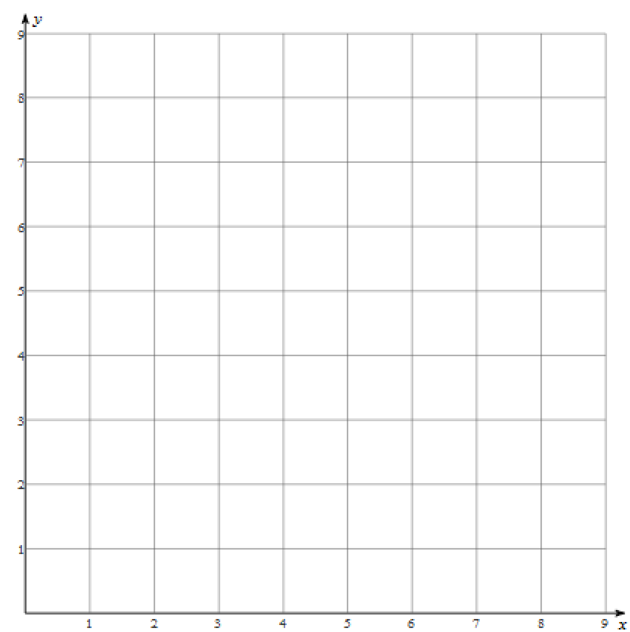
Determine the *minimum* value of A = 5x + 4y for the given constraints.

$5x + 2y \ge 14$

$2x + 3y \ge 10$

$x \ge 0$

$y \ge 0$



STEP 1: Graph the constraints and shade the feasible region.

STEP 2: Calculate the value of A at each vertex of the feasible region.

(x, y)	A = 5x + 4y

The *minimum* value of A is _____, which occurs at _______.

Notice that the feasible region is *unbounded* and there is *no maximum* value.

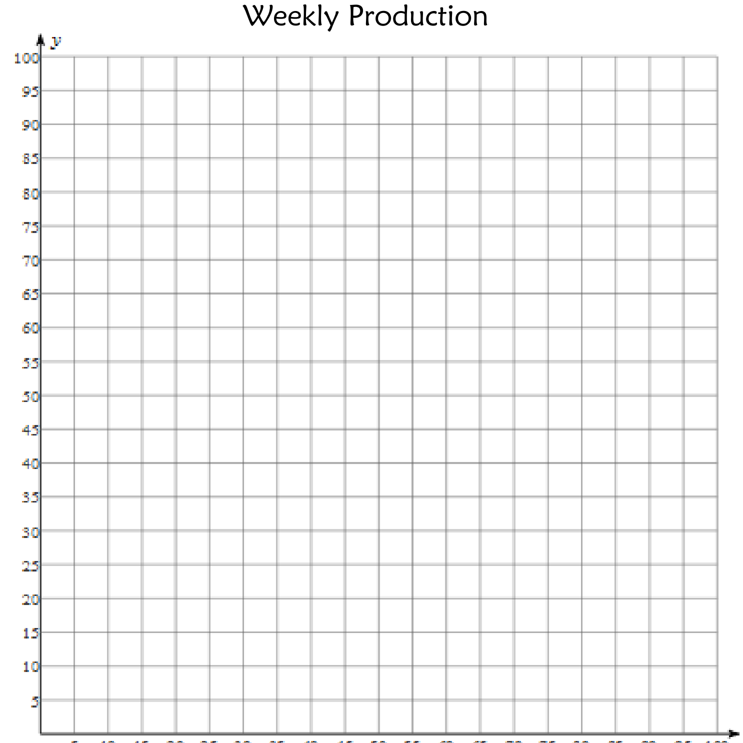
Example 3: Finding the Maximum Profit

Cam's Company produces and sells two products. The company makes a profit of $24 on each thingamajig sold and $18 on each doohickey sold. The company can produce no more than 72 thingamajigs per week and no more than 90 doohickies per week. In addition, the company can produce no more than a total of 140 thingamajigs and doohickies in one week. What is the maximum profit that the company can make in one week, and how many thingamajigs and doohickies would have to be produced and sold for this maximum profit?

STEP 1: Define the variables and write the constraints.

STEP 2: Graph the constraints and shade the feasible region.

Weekly Production



STEP 3: Write the objective (profit) function.

STEP 4: Calculate the value of the profit function at each vertex of the feasible region and look for the maximum profit.

(x, y)	

The company can make a maximum profit of ___________ in a week by producing and selling _______ thingamajigs and _______ doohickies.

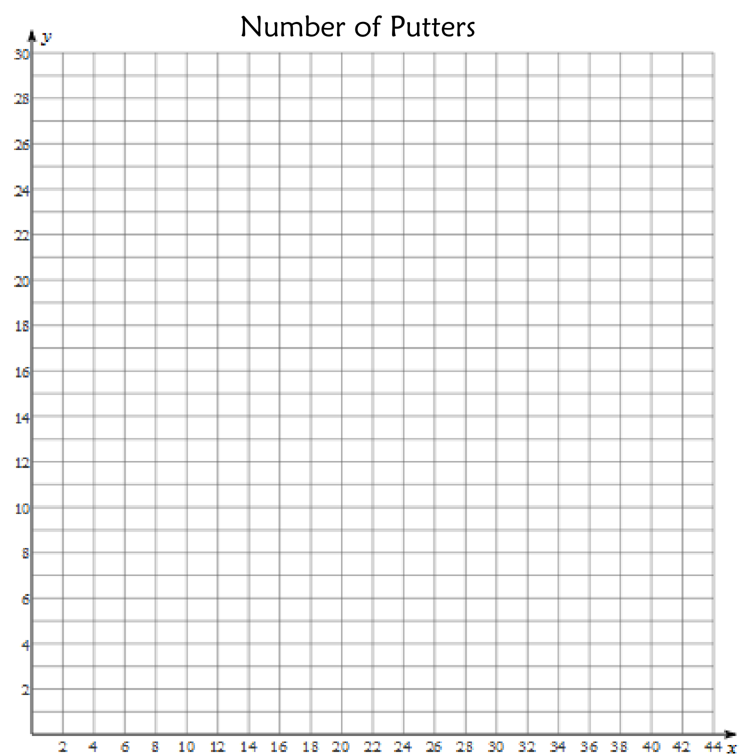
Example 4: Finding the Maximum Profit

A golf club manufactures two types of putters; a mallet and a blade. The mallet needs 2 hours in the assembly room and 0.5 hour in the finishing room. The blade model needs 3 hours in the assembly room and 1.5 hours in the finishing room. The assembly room is available for 84 hours a week, and the finishing room is available for 36 hours a week. The profit for each mallet sold is $18 and the profit for each blade sold is $45. How many of each type of putter should be produced and sold to maximize profit?

STEP 1: Organize the information in a table.

	Mallet	Blade	Time available (h)
assembly room (h)			
finishing room (h)			
profit ($)			

STEP 2: Define the variables and write the constraints.



STEP 3: Graph the constraints and shade the feasible region.

STEP 4: Write the objective (profit) function.

STEP 5: Calculate the value of the profit function at each vertex of the feasible region and look for the maximum profit.

(x, y)	

The maximum weekly profit of __________ occurs by producing and selling ______ mallets and ______ blades.

Suppose the profit for each blade increases to $55. How many of each type of putter should the company produce to maximize profit?

The feasible region does not change, so we need only revise the profit function.

(x, y)	

The maximum weekly profit of __________ occurs by producing and selling ______ mallets and ______ blades.

Example 5: Finding the Minimum Cost

A furnace supplier needs to send at least 80 furnaces from its distribution centre to its two warehouses. The north warehouse has room for 60 furnaces. The south warehouse has room for 45 furnaces. It costs $20 to send a furnace to the north warehouse and $16 to send a furnace to the south warehouse. How many furnaces should be shipped to each warehouse to minimize cost? What is the minimum cost?

STEP 1: Define the variables and write the constraints.

STEP 2: Graph the constraints and shade the feasible region.

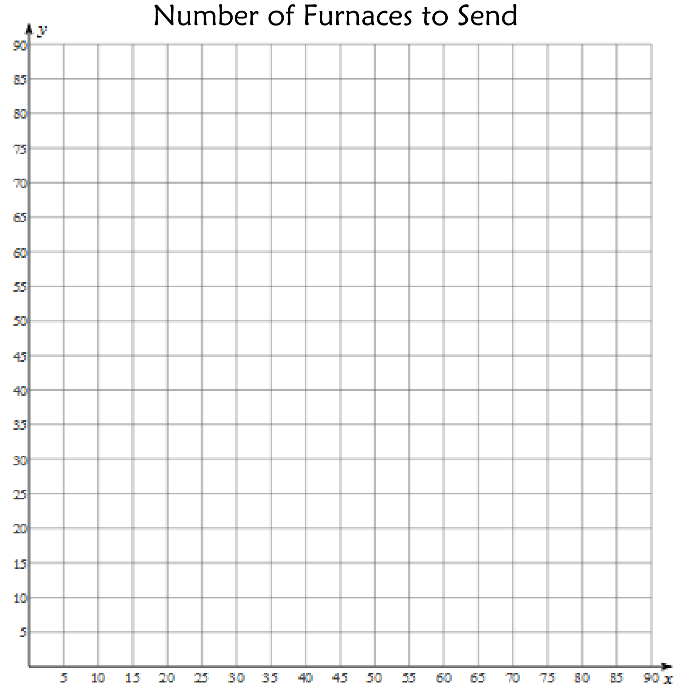
STEP 3: Write the objective (cost) function.

STEP 4: Calculate the value of the cost function at each vertex of the feasible region and look for the minimum cost.

(x, y)	

The supplier can minimize the cost by sending ______ furnaces to the north warehouse and ______ to the south warehouse, for a minimum cost of _________.

Number of Furnaces to Send



Suppose the cost is actually found to be $20 to send a furnace to either warehouse. How many furnaces should be sent to each warehouse to minimize cost?

The feasible region does not change, so we need only revise the cost function.

(x, y)	

The points __________ and __________ both give the minimum cost of __________. Therefore, any whole number coordinates on the line segment between these points would also give this min. cost. The points __________, __________, __________, … , __________, __________ all represent solutions.

Of the 80 furnaces the supplier will send, any number from ______ to ______ can be sent to the north warehouse, and the rest can be sent to the south warehouse.

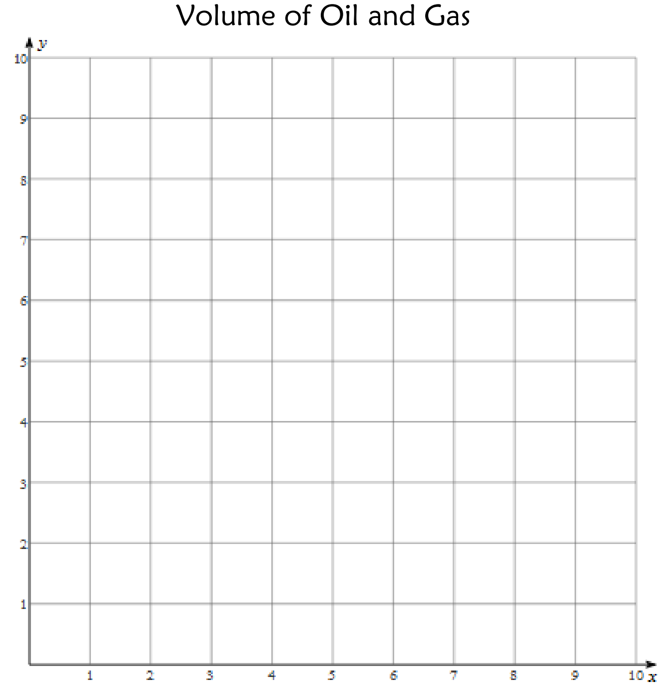
Example 6: Finding the Maximum Revenue

A refinery produces oil and gas. The refinery can produce up to 9 million litres of heating oil and 6 million litres of gasoline each day. Based on demand, at least 2 litres of gasoline is produced for each litre of heating oil. Heating oil is projected to sell for $1.75 per litre, and gasoline is projected to sell for $1.10 per litre. How many litres of oil and gas should be produced each day in order to maximize revenue? What is the maximum revenue?

STEP 1: Define the variables and write the constraints.

STEP 2: Graph the constraints and shade the feasible region.

Volume of Oil and Gas



STEP 3: Write the objective (revenue) function.

STEP 4: Calculate the value of the cost function at each vertex of the feasible region and look for the maximum revenue.

(x, y)	

The refinery can maximize its revenue by producing ___________ litres of oil and___________ litres of gasoline each day. The maximum daily revenue would be ________________________.