Trigonometric Identities

In mathematics, it is important to understand the difference between an *equation* and an *identity*.

- An *equation* is only true for *certain* value(s) of the variable. For example, the equation $x^2 - 4 = 0$ is true *only* for $x = \pm 2$.
- An *identity* is an equation that is true for *all* values of the variable. For example, the equation $(x-1)^2 = x^2 - 2x + 1$ is true for *all* values of x, so we can call it an identity.

A *trigonometric* identity is a trigonometric equation that is true for all *permissible* values of the variable.

- An example of a *trigonometric equation* is $\sin x = 1$, which is true only for $x = 90^o \pm 360^o n,\ n \in W$.
- An example of a *trigonometric identity* is $\sin^2 x + \cos^2 x = 1$, which is true for *all* values of x.
- An example of a trigonometric identity with *non-permissible* values is $\csc x = \dfrac{1}{\sin x}$, which is true for all values of x except $x = 0^o \pm 180^o n,\ n \in W$, since $\sin x = 0$ for these values of x.

RECIPROCAL AND QUOTIENT IDENTITES

We are already familiar with a few trigonometric identities, referred to as the *reciprocal* and *quotient* identities.

RECIPROCAL IDENTITIES:

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

QUOTIENT IDENTITIES:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

We will be using these and other *fundamental* trigonometric identities to prove *other* trigonometric identities.

Example 1: Verify, Prove, and State Non-Permissible Values for a Trigonometric Identity

Consider the equation $\sec\theta = \dfrac{\tan\theta}{\sin\theta}$.

a. **Verify that the equation** $\sec\theta = \dfrac{\tan\theta}{\sin\theta}$ **is valid for** $\theta = 60°$.

LHS = $\sec\theta = \sec 60^o =$

RHS = $\dfrac{\tan\theta}{\sin\theta} = \dfrac{\tan 60^o}{\sin 60^o} =$

Therefore, the equation is true for $\theta = 60°$.

Note: *Verifying that an equation is valid for a particular value of the variable is not sufficient to conclude that the equation is an identity.*

b. **Prove that** $\sec\theta = \dfrac{\tan\theta}{\sin\theta}$ **is an *identity*.**

Note: *To prove that an identity is true for all permissible values, express both sides of the identity in equivalent forms. One or both sides of the identity must be algebraically manipulated into an equivalent form to match the other side.*

There is a major difference between solving a trigonometric equation and proving a trigonometric identity:

- *Solving* a trigonometric equation determines the value(s) that make that particular equation true. You perform operations across the = sign to solve for the variable.
- *Proving* a trigonometric identity shows that the expressions on each side of the = sign are equivalent. You work on each side of the identity *independently*, and you do *not* perform operations across the = sign.

LHS	RHS
$\sec\theta$	$\dfrac{\tan\theta}{\sin\theta}$

c. **Determine the non-permissible values, in degrees, for the identity** $\sec\theta = \dfrac{\tan\theta}{\sin\theta}$.

Note: Non-permissible values *of the variable are values for which* ***denominators are zero*** *or* ***numerators or denominators are undefined.***

To determine the non-permissible values, assess each trigonometric function in the equation individually.

LHS: $\sec\theta$ is undefined when when θ = ______________________________.

RHS: $\tan\theta$ is undefined when θ = ______________________________.

$\sin\theta = 0$ when θ = ______________________________.

Combined, the non-permissible values for are: θ = ____.

Example 2: Use Identities to Simplify Expressions

Simplify the expression $\dfrac{\tan x \cos x}{\sec x \cot x}$.

To simplify the expression, we can use the reciprocal and quotient identities to rewrite the expression in terms of sine and cosine.

$$\frac{\tan x \cos x}{\sec x \cot x} =$$

Example 3: Prove Identities

Prove each of the following identities:

$\dfrac{\tan x \cos x}{\sin x} = 1$	
LHS	**RHS**
$\dfrac{\tan x \cos x}{\sin x}$	1

$\cot x + \sec x = \dfrac{\cos^2 x + \sin x}{\sin x \cos x}$	
LHS	**RHS**
$\cot x + \sec x$	$\dfrac{\cos^2 x + \sin x}{\sin x \cos x}$

PYTHAGOREAN IDENTITIES

There are three forms of the Pythagorean identity:

1) Derived from the Pythagorean Theorem, we know that $x^2 + y^2 = r^2$.

   In the unit circle, $x = \cos\theta$, and $y = \sin\theta$ and r = 1. Substitute these values for x, y and r:

2) Divide both sides of $\cos^2\theta + \sin^2\theta = 1$ by $\sin^2\theta$ and apply the reciprocal and quotient identities:

3) Divide both sides of $\cos^2\theta + \sin^2\theta = 1$ by $\cos^2\theta$ and apply the reciprocal and quotient identities:

PYTHAGOREAN IDENTITES:

$\cos^2\theta + \sin^2\theta = 1$ $\qquad \cot^2\theta + 1 = \csc^2\theta$ $\qquad 1 + \tan^2\theta = \sec^2\theta$

Example 4: Use the Pythagorean Identities

Prove each of the following identities:

a. $\cos x - \sin^2 x \cos x = \cos^3 x$ $\qquad$ b. $\sin x = \dfrac{\sec x}{\cot x + \tan x}$ $\qquad$ c. $\tan x \sin x = \sec x - \cos x$

a. $\cos x - \sin^2 x \cos x = \cos^3 x$	
LHS	**RHS**
$\cos x - \sin^2 x \cos x$	$\cos^3 x$

b. $\sin x = \dfrac{\sec x}{\cot x + \tan x}$	
LHS	**RHS**
$\sin x$	$\dfrac{\sec x}{\cot x + \tan x}$

c. $\tan x \sin x = \sec x - \cos x$	
LHS	**RHS**
$\tan x \sin x$	$\sec x - \cos x$

Tips for Proving Trigonometric Identities:

- It's easier to simplify a more complicated expression than to make a simple expression more complicated, so start with the more complicated side of the identity.
- Use known identities to make substitutions.
- Rewrite the expressions using sine and cosine only.
- If a quadratic is present, consider the Pythagorean identities.
- Factor expressions to cancel out terms or create other identities.
- Expand expressions to create identities.
- If you have fractions, think about making common denominators.
- Multiply the numerator and denominator by the conjugate of an expression.
- Always keep the "target expression" in mind. Continually refer to it.

Example 5: Solve Equations Using the Quotient, Reciprocal, and Pythagorean Identities

Solve for θ in general form and then in the specified domain. Use exact solutions when possible. Check for non-permissible values.

a. $\cos^2\theta = \cot\theta\sin\theta,\ 0 \le \theta < 2\pi$
b. $2\sin\theta = 7 - 3\csc\theta,\ -360° < \theta < 180°$
c. $\sin^2\theta + 2\cos\theta = 2,\ -2\pi \le \theta < 2\pi$

Solution:

a. $\cos^2\theta = \cot\theta\sin\theta,\ 0 \le \theta < 2\pi$

b. $2\sin\theta = 7 - 3\csc\theta,\ -360° < \theta < 180°$

c. $\sin^2\theta + 2\cos\theta = 2,\ -2\pi \le \theta < 2\pi$