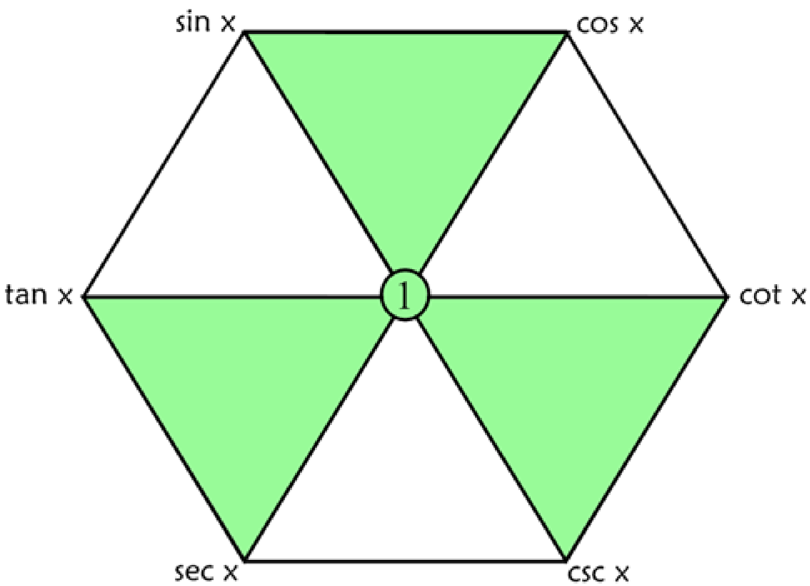
sin x
cos x
tan x
1
cot x
sec x
csc x

How could you calculate $\cos 15^{\circ}$ *without* a calculator?

Sum, Difference, and Double-Angle Identities

The sum and difference identities are used to simplify expressions and to determine the exact trigonometric values of some angles.

SUM IDENTITIES	Examples:
$\sin(A+B)=\sin A\cos B+\cos A\sin B$ $\cos(A+B)=\cos A\cos B-\sin A\sin B$ $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$	$\sin(12^\circ+23^\circ)=\sin 12^\circ\cos 23^\circ+\cos 12^\circ\sin 23^\circ$ $\cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\cos\frac{\pi}{6}\cos\frac{\pi}{4}-\sin\frac{\pi}{6}\sin\frac{\pi}{4}$ $\tan(40^\circ+25^\circ)=\dfrac{\tan 40^\circ+\tan 25^\circ}{1-\tan 40^\circ\tan 25^\circ}$
DIFFERENCE IDENTITIES	**Examples:**
$\sin(A-B)=\sin A\cos B-\cos A\sin B$ $\cos(A-B)=\cos A\cos B+\sin A\sin B$ $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$	$\sin(52^\circ-33^\circ)=\sin 52^\circ\cos 33^\circ-\cos 52^\circ\sin 33^\circ$ $\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{3}\sin\frac{\pi}{4}$ $\tan(70^\circ-35^\circ)=\dfrac{\tan 70^\circ-\tan 35^\circ}{1+\tan 70^\circ\tan 35^\circ}$

Example 1: Simplify Expressions Using Sum and Difference Identities

Write each expression as a single trigonometric function and determine the exact value for each expression.

a. $\cos 290° \cos 80° + \sin 290° \sin 80°$

b. $\sin\left(\frac{5\pi}{12}\right)\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{5\pi}{12}\right)\sin\left(\frac{\pi}{3}\right)$

Solution:

$\cos 290° \cos 80° + \sin 290° \sin 80°$	$\sin\left(\frac{5\pi}{12}\right)\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{5\pi}{12}\right)\sin\left(\frac{\pi}{3}\right)$

Example 2: Determine Exact Trigonometric Values for Angles

Find the exact value for each of the following:

a. $\sin(195°)$ b. $\cos\left(\frac{5\pi}{12}\right)$

Solution:

$\sin(195°)$	$\cos\left(\frac{5\pi}{12}\right)$
	$=\cos\left(\frac{2\pi}{12}+\frac{3\pi}{12}\right)$ $=\cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right)$

$$=\cos\frac{\pi}{6}\cos\frac{\pi}{4}-\sin\frac{\pi}{6}\cdot\sin\frac{\pi}{4}$$

$$=\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}$$

$$=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}$$

$$=\boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

Example 3: Prove an Identity Using the Compound Angle Identities

Prove each of the following identities.

a. $\sin\left(\frac{\pi}{2} - x\right) = \cos x$

b. $\sin\left(x + \frac{\pi}{3}\right) - \cos\left(x + \frac{\pi}{6}\right) = \sin x$

c. $\frac{\sin(a + b)}{\cos a \cos b} = \tan a + \tan b$

Solution:

a.

$\sin\left(\frac{\pi}{2} - x\right) = \cos x$	
LHS	**RHS**
$\sin\left(\frac{\pi}{2} - x\right)$	$\cos x$

b.

$\sin\left(x+\frac{\pi}{3}\right)-\cos\left(x+\frac{\pi}{6}\right)=\sin x$	
LHS	**RHS**
$\sin\left(x+\frac{\pi}{3}\right)-\cos\left(x+\frac{\pi}{6}\right)$	$\sin x$

$$= \sin x \cos\frac{\pi}{3} + \sin\frac{\pi}{3}\cos x - \left(\cos x \cos\frac{\pi}{6} - \sin x \sin\frac{\pi}{6}\right)$$

$$= \sin x \cdot \left(\frac{1}{2}\right) + \cancel{\left(\frac{\sqrt{3}}{2}\right)\cos x} - \cancel{\cos x\left(\frac{\sqrt{3}}{2}\right)} + \sin x\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\sin x + \frac{1}{2}\sin x = \sin x$$

$\frac{\sin(a+b)}{\cos a \cos b} = \tan a + \tan b$	
LHS	**RHS**
$\frac{\sin(a+b)}{\cos a \cos b}$	$\tan a + \tan b$

$$= \frac{\sin a \cos b + \sin b \cos a}{\cos a \cos b}$$

$$= \frac{\sin a \cancel{\cos b}}{\cos a \cancel{\cos b}} + \frac{\sin b \cancel{\cos a}}{\cancel{\cos a}\cos b}$$

$$= \frac{\sin a}{\cos a} + \frac{\sin b}{\cos b}$$

$$= \tan a + \tan b$$

Example 4: Solve an Equation Involving the Compound Angle Identities

Solve $\sin x \cos\frac{\pi}{6} - \cos x \sin\frac{\pi}{6} = \frac{1}{2}, 0 \le x \le 2\pi$

Solution:

$\sin x \cos\frac{\pi}{6} - \cos x \sin\frac{\pi}{6} = \frac{1}{2}, 0 \le x \le 2\pi$

$= \sin\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$

$$x - \frac{\pi}{6} = \begin{cases} \frac{\pi}{6} \pm 2\pi n \\ \frac{5\pi}{6} \pm 2\pi n \end{cases}, n \in W$$

$$x = \begin{cases} \frac{\pi}{3} \pm 2\pi n \\ \pi \pm 2\pi n \end{cases}, n \in W$$

$\therefore \boxed{x = \frac{\pi}{3}, \pi}$

Example 5: Using Compound Angle Identities

If $\cos\theta = \frac{-3}{5}$ and $\frac{\pi}{2} < \theta < \pi$, determine the value of $\sin\left(\theta + \frac{\pi}{6}\right)$.

Solution:

DOUBLE ANGLE IDENTITIES	Examples:
$\sin 2A = 2\sin A\cos A$	$\sin\frac{\pi}{4} = 2\sin\frac{\pi}{8}\cos\frac{\pi}{8}$
$\cos 2A = \cos^2 A - \sin^2 A$ $\cos 2A = 2\cos^2 A - 1$ $\cos 2A = 1 - 2\sin^2 A$	$\cos 140° = \cos^2 70° - \sin^2 70°$ $\cos 140° = 2\cos^2 70° - 1$ $\cos 140° = 1 - 2\sin^2 70°$
$\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$	$\tan\frac{\pi}{6} = \dfrac{2\tan\frac{\pi}{12}}{1-\tan^2\frac{\pi}{12}}$

Example ~~5~~ 6: Simplify Expressions Using Double-Angle Identities

Write each expression as a single trigonometric function and determine the exact value for each expression.

a. $\cos^2 \frac{\pi}{3} - \sin^2 \frac{\pi}{3}$

b. $2 \sin \frac{\pi}{12} \cos \frac{\pi}{12}$

Solution:

a.	b.
$\cos^2 \frac{\pi}{3} - \sin^2 \frac{\pi}{3}$	$2 \sin \frac{\pi}{12} \cos \frac{\pi}{12}$

Example ~~6~~ 7: Simplify Expressions Using Identities & Recognizing Non-Permissible Values

Consider the expression $\dfrac{1-\cos 2x}{\sin 2x}$

a. Determine the non-permissible values for the expression.
b. Simplify the expression.

Solution:

a. $\sin 2x \neq 0$ This means that $2\sin x\cos x \neq 0$ so $\sin x \neq 0$ $\cos x \neq 0$

$\sin x = 0$ when $x = 0 \pm \pi n,\ n \in W$

$\cos x = 0$ *when* $x = \dfrac{\pi}{2} \pm \pi n,\ n \in W$

Combining these values means $x \neq \pm\dfrac{\pi}{2}n,\ n \in W$

b. $\dfrac{1-\cos 2x}{\sin 2x}$

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Example 8: Prove an Identity Using a Double Angle Identity

Prove the following identities.

a. $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$

b. $\dfrac{\cos 2x}{\cos x - \sin x} = \cos x + \sin x$

c. $\tan 2x - 2\tan 2x \sin^2 x = \sin 2x$

d. $\dfrac{1-\cos 2x}{\sin x \sin 2x} = \sec x$

a. $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$

$\dfrac{\sin 2x}{1+\cos 2x} = \tan x$	
LHS	**RHS**
$\dfrac{\sin 2x}{1+\cos 2x}$	$\tan x$

b. $\dfrac{\cos 2x}{\cos x - \sin x} = \cos x + \sin x$

$\dfrac{\cos 2x}{\cos x - \sin x} = \cos x + \sin x$	
LHS	**RHS**
$\dfrac{\cos 2x}{\cos x - \sin x}$	$\cos x + \sin x$

c. $\tan 2x - 2\tan 2x \sin^2 x = \sin 2x$

$\tan 2x - 2\tan 2x \sin^2 x = \sin 2x$	
LHS	**RHS**
$\tan 2x - 2\tan 2x \sin^2 x$	$\sin 2x$

d. $\dfrac{1-\cos 2x}{\sin x \sin 2x} = \sec x$

$\dfrac{1-\cos 2x}{\sin x \sin 2x} = \sec x$	
LHS	**RHS**
$\dfrac{1-\cos 2x}{\sin x \sin 2x}$	$\sec x$

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Example [redacted] 9: Solving Equations Using the Double Angle Identity

Solve for θ in general form and then in the specified domain. Use exact solutions when possible.

a. $\cos 2\theta + 1 - \cos\theta = 0,\ 0 \le \theta < 2\pi$
b. $\sin 2\theta + \sin\theta = 6\cos\theta + 3,\quad -180° \le \theta \le 180°$
c. $3\tan\theta = \tan 2\theta,\ 0 \le \theta < 2\pi$

Solution:

a. $\cos 2\theta + 1 - \cos\theta = 0,\ 0 \le \theta < 2\pi$

b. $\sin 2\theta + \sin\theta = 6\cos\theta + 3; -180° \le \theta \le 180°$

c. $3\tan\theta = \tan 2\theta,\ 0 \le \theta < 2\pi$