Sum, Difference, and Double-Angle Identities

The sum and difference identities are used to simplify expressions and to determine the exact trigonometric values of some angles.

SUM IDENTITIES	Examples:
$\sin(A+B)=\sin A\cos B+\cos A\sin B$ $\cos(A+B)=\cos A\cos B-\sin A\sin B$ $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$	$\sin(12°+23°)=\sin 12°\cos 23°+\cos 12°\sin 23°$ $\cos\left(\dfrac{\pi}{6}+\dfrac{\pi}{4}\right)=\cos\dfrac{\pi}{6}\cos\dfrac{\pi}{4}-\sin\dfrac{\pi}{6}\sin\dfrac{\pi}{4}$ $\tan(40°+25°)=\dfrac{\tan 40°+\tan 25°}{1-\tan 40°\tan 25°}$
DIFFERENCE IDENTITIES	**Examples:**
$\sin(A-B)=\sin A\cos B-\cos A\sin B$ $\cos(A-B)=\cos A\cos B+\sin A\sin B$ $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$	$\sin(52°-33°)=\sin 52°\cos 33°-\cos 52°\sin 33°$ $\cos\left(\dfrac{\pi}{3}-\dfrac{\pi}{4}\right)=\cos\dfrac{\pi}{3}\cos\dfrac{\pi}{4}+\sin\dfrac{\pi}{3}\sin\dfrac{\pi}{4}$ $\tan(70°-35°)=\dfrac{\tan 70°-\tan 35°}{1+\tan 70°\tan 35°}$

Example 1: Simplify Expressions Using Sum and Difference Identities

Write each expression as a single trigonometric function and determine the exact value for each expression.

a. $\cos 290°\cos 80°+\sin 290°\sin 80°$

b. $\sin\left(\dfrac{5\pi}{12}\right)\cos\left(\dfrac{\pi}{3}\right)+\cos\left(\dfrac{5\pi}{12}\right)\sin\left(\dfrac{\pi}{3}\right)$

Solution:

$\cos 290°\cos 80°+\sin 290°\sin 80°$	$\sin\left(\dfrac{5\pi}{12}\right)\cos\left(\dfrac{\pi}{3}\right)+\cos\left(\dfrac{5\pi}{12}\right)\sin\left(\dfrac{\pi}{3}\right)$

Example 2: Determine Exact Trigonometric Values for Angles

Find the exact value for each of the following:

$\sin(195°)$	$\cos\left(\frac{5\pi}{12}\right)$

Example 3: Prove an Identity Using the Compound Angle Identities

Prove each of the following identities.

a. $\sin\left(\frac{\pi}{2} - x\right) = \cos x$

LHS	RHS
$\sin\left(\frac{\pi}{2} - x\right)$	$\cos x$

b. $\sin\left(x + \frac{\pi}{3}\right) - \cos\left(x + \frac{\pi}{6}\right) = \sin x$

LHS	RHS
$\sin\left(x + \frac{\pi}{3}\right) - \cos\left(x + \frac{\pi}{6}\right)$	$\sin x$

c. $\dfrac{\sin(a+b)}{\cos a \cos b} = \tan a + \tan b$

LHS	RHS
$\dfrac{\sin(a+b)}{\cos a \cos b}$	$\tan a + \tan b$

Example 4: Solve an Equation Involving the Compound Angle Identities

Solve $\sin x \cos\dfrac{\pi}{6} - \cos x \sin\dfrac{\pi}{6} = \dfrac{1}{2}, 0 \le x \le 2\pi$

Solution:

$$\sin x \cos\frac{\pi}{6} - \cos x \sin\frac{\pi}{6} = \frac{1}{2}, 0 \le x \le 2\pi$$

Example 5: Using Compound Angle Identities

If $\cos\theta = \dfrac{-3}{5}$ and $\dfrac{\pi}{2} < \theta < \pi$, determine the value of $\sin\left(\theta + \dfrac{\pi}{6}\right)$.

Solution:

DOUBLE ANGLE IDENTITIES	Examples:
$\sin 2A = 2\sin A\cos A$	$\sin\frac{\pi}{4} = 2\sin\frac{\pi}{8}\cos\frac{\pi}{8}$
$\cos 2A = \cos^2 A - \sin^2 A$	$\cos 140° = \cos^2 70° - \sin^2 70°$
$\cos 2A = 2\cos^2 A - 1$	$\cos 140° = 2\cos^2 70° - 1$
$\cos 2A = 1 - 2\sin^2 A$	$\cos 140° = 1 - 2\sin^2 70°$
$\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$	$\tan\frac{\pi}{6} = \dfrac{2\tan\frac{\pi}{12}}{1-\tan^2\frac{\pi}{12}}$

Example 6: Simplify Expressions Using Double-Angle Identities

Write each expression as a single trigonometric function and determine the exact value for each expression.

a. $\cos^2\frac{\pi}{3} - \sin^2\frac{\pi}{3}$

b. $2\sin\frac{\pi}{12}\cos\frac{\pi}{12}$

Solution:

$\cos^2\frac{\pi}{3} - \sin^2\frac{\pi}{3}$	$2\sin\frac{\pi}{12}\cos\frac{\pi}{12}$

Example 7: Simplify Expressions Using Identities & Recognizing Non-Permissible Values

Consider the expression $\dfrac{1-\cos 2x}{\sin 2x}$

a. Determine the non-permissible values for the expression.
b. Simplify the expression.

Solution:

a. $\sin 2x \neq 0$ This means that $2\sin x\cos x \neq 0$ So, $\sin x \neq 0$ $\cos x \neq 0$ $\sin x = 0$ when $x = 0 \pm \pi n,\ n \in W$ $\cos x = 0$ *when* $x = \dfrac{\pi}{2} \pm \pi n,\ n \in W$ Combining these values means $x \neq \pm\dfrac{\pi}{2}n,\ n \in W$	b. $\dfrac{1-\cos 2x}{\sin 2x}$

Example 8: Prove an Identity Using a Double Angle Identity

Prove the following identities.

a. $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$

b. $\dfrac{\cos 2x}{\cos x - \sin x} = \cos x + \sin x$

c. $\tan 2x - 2\tan 2x \sin^2 x = \sin 2x$

d. $\dfrac{1-\cos 2x}{\sin x \sin 2x} = \sec x$

Solution:

a. $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$

$\dfrac{\sin 2x}{1+\cos 2x} = \tan x$	
LHS	**RHS**
$\dfrac{\sin 2x}{1+\cos 2x}$	$\tan x$

b. $\dfrac{\cos 2x}{\cos x - \sin x} = \cos x + \sin x$

$\dfrac{\cos 2x}{\cos x - \sin x} = \cos x + \sin x$	
LHS	**RHS**
$\dfrac{\cos 2x}{\cos x - \sin x}$	$\cos x + \sin x$

c. $\tan 2x - 2\tan 2x \sin^2 x = \sin 2x$

$\tan 2x - 2\tan 2x \sin^2 x = \sin 2x$	
LHS	**RHS**
$\tan 2x - 2\tan 2x \sin^2 x$	$\sin 2x$

d. $\dfrac{1-\cos 2x}{\sin x \sin 2x} = \sec x$

$\dfrac{1-\cos 2x}{\sin x \sin 2x} = \sec x$	
LHS	**RHS**
$\dfrac{1-\cos 2x}{\sin x \sin 2x}$	$\sec x$

Example 9: Solving Equations Using the Double Angle Identity

Solve for θ in general form and then in the specified domain. Use exact solutions when possible.

a. $\cos 2\theta + 1 - \cos\theta = 0,\ 0 \le \theta < 2\pi$
b. $\sin 2\theta + \sin\theta = 6\cos\theta + 3,\quad -180° \le \theta \le 180°$
c. $3\tan\theta = \tan 2\theta,\ 0 \le \theta < 2\pi$

Solution:

a. $\cos 2\theta + 1 - \cos\theta = 0,\ 0 \le \theta < 2\pi$

b. $\sin 2\theta + \sin\theta = 6\cos\theta + 3; -180° \le \theta \le 180°$

c. $3\tan\theta = \tan 2\theta,\ 0 \le \theta < 2\pi$