Analyzing Rational Functions

VERTICAL ASYMPTOTES AND POINTS OF DISCONTINUITY

Graphs of rational functions have a variety of shapes and features. We have seen that the graphs of rational functions, $f(x)=\frac{p(x)}{q(x)}$, are *discontinuous* for any non-permissible values of x (values of x that make $q(x)=0$).
One of the possible features that correspond to a non-permissible value of x is a ***vertical asymptote***. Another possible feature that corresponds to a non-permissible value of x is a ***point of discontinuity***.

Point of Discontinuity

- A point, described by an ordered pair, at which the graph of the function is not continuous.
- Results in a single point missing from the graph, which is represented by an open circle.

Example 1: Graph a Rational Function with a Point of Discontinuity

Sketch the graph of $f(x)=\frac{x^2-3x-4}{x-4}$. Analyze its behaviour near its non-permissible value.

Solution:

Factor the numerator and denominator, then simplify. $f(x)=\frac{x^2-3x-4}{x-4}=$

The non-permissible value of x is x = ______.

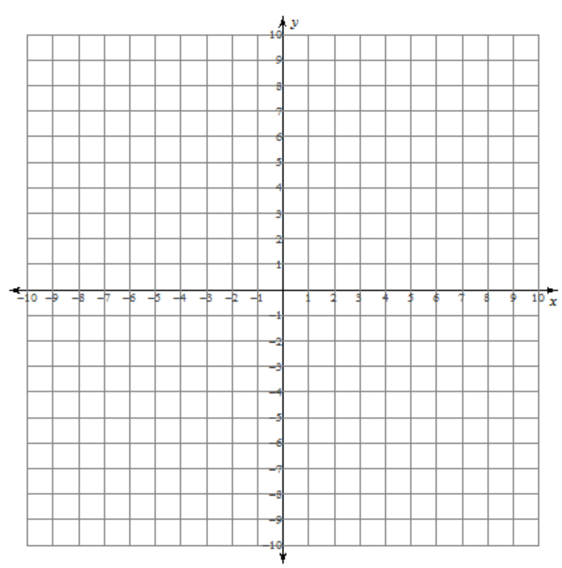
Note that the simplified function is a ____________________ function.

Complete the table of values to observe the function's behaviour near its non-permissible value of x=4:

x	3.5	3.9	3.99	3.999	4	4.001	4.01	4.1	4.5
f(x)					DNE				

From the table we can see that the value of f(x) gets closer and closer to _____ as x gets closer to 4 from *either* side. In other words, $\lim_{x\to 4} f(x)=5$.

- Recall that for any *rational* function, f(x), if f(x) approaches $\pm\infty$ as x approaches (from either side) a non-permissible value, then there will be a *vertical asymptote* at that non-permissible value.
- If f(x) approaches (from either side) a finite *limit* as x approaches a non-permissible value, then there will be a *point of discontinuity* at that non-permissible value of x.



So, for this example, there is a ____________________ at (_____, _____) *

* To easily determine the y-coordinate of the POD, substitute the x-coordinate of the POD into the *simplified* function and evaluate:

Determining Vertical Asymptotes and Points of Discontinuity

To quickly determine if the graph of a rational function will have a vertical asymptote, a point of discontinuity, or both, you begin by factoring the numerator and denominator. There will be a:

- **Vertical Asymptote** if the factor that corresponds to the non-permissible value appears in the denominator to a *greater* degree than in the numerator.

- **Point of Discontinuity** if the factor that corresponds to the non-permissible value appears in the denominator to an *equal or lesser* degree than in the numerator.

- **Examples:**

$f(x)=\frac{(x-1)(x+3)}{(x+1)}$ has a __________ at __________

$g(x)=\frac{(x+2)}{(x+2)(x+2)}$ has a __________ at __________

$h(x)=\frac{(x-1)(x+4)}{(x+4)}$ has a __________ at __________

$k(x)=\frac{(x+2)(x-3)}{(x+1)(x-3)}$ has a __________ at __________ and a __________ at __________

HORIZONTAL ASYMPTOTES

To determine whether or not a rational function has a horizontal asymptote, consider the *end behaviour* of the function (that is, the behaviour of f(x) as x approaches $\pm\infty$).

- If f(x) approaches $\pm\infty$, then there is no horizontal asymptote.

- If f(x) approaches a constant, c, then there is a horizontal asymptote at y = c.

To help us determine the end behavior of the function, and, subsequently the equation of the horizontal asymptote, if it exists, we can divide both *numerator* and *denominator* by the highest power of x that appears in the expression. Then, simplify this expression and examine the behavior of the function as x approaches $\pm\infty$. In other words, evaluate the limit of the function as $x\rightarrow\pm\infty$.

Examples:

Determine the equation of the horizontal asymptote, if it exists, for each function:

$f(x)=\frac{8x^3+2x+3}{4x^3+5x^2}$ Horizontal asymptote: ____________________

$g(x)=\frac{2x^2+3x}{4x^3-1}$ Horizontal asymptote: ____________________

$h(x)=\frac{x^3+x+9}{2x^2+1}$ Horizontal asymptote: ____________________

This procedure can be simplified to quickly determine the equation of the horizontal asymptote:

- If the numerator and denominator have the ***same*** degree and the leading coefficients are, respectively, *a* and *b*, then the horizontal asymptote is given by $y=\frac{a}{b}$.

  (For $f(x)=\frac{8x^3+2x+3}{4x^3+5x^2}$ the horizontal asymptote is at y = 2)

- If the degree of the ***denominator*** is greater than that of the numerator, then the horizontal asymptote is given by y = 0.

  (For $g(x)=\frac{2x^2+3x}{4x^3-1}$ the horizontal asymptote is at y = 0)

- If the degree of the ***numerator*** is greater than that of the denominator, then the function will approach $\pm\infty$ and there will be ***no*** horizontal asymptote.

  (For $h(x)=\frac{x^3+x+9}{2x^2+1}$ there is *no* horizontal asymptote)

Example 2: Sketch the Graph of a Rational Function

Sketch the graphs of the following functions:

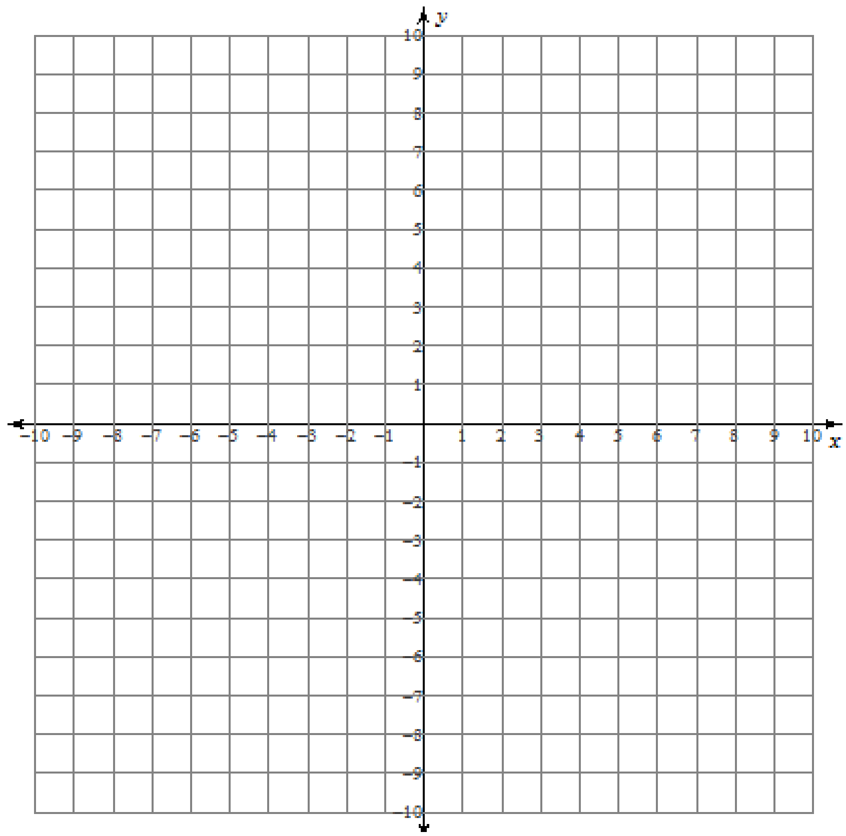
a. $f(x) = \dfrac{x^2 - 3x + 2}{x - 1}$ b. $f(x) = \dfrac{x - 1}{x^2 - x - 2}$ c. $f(x) = \dfrac{x^2 + 2x - 8}{x^2 + 5x + 4}$

Solution:

Express each function in factored form. Sketch the graphs using x-and y-intercepts, asymptotes, points of discontinuity, and any other necessary key points.

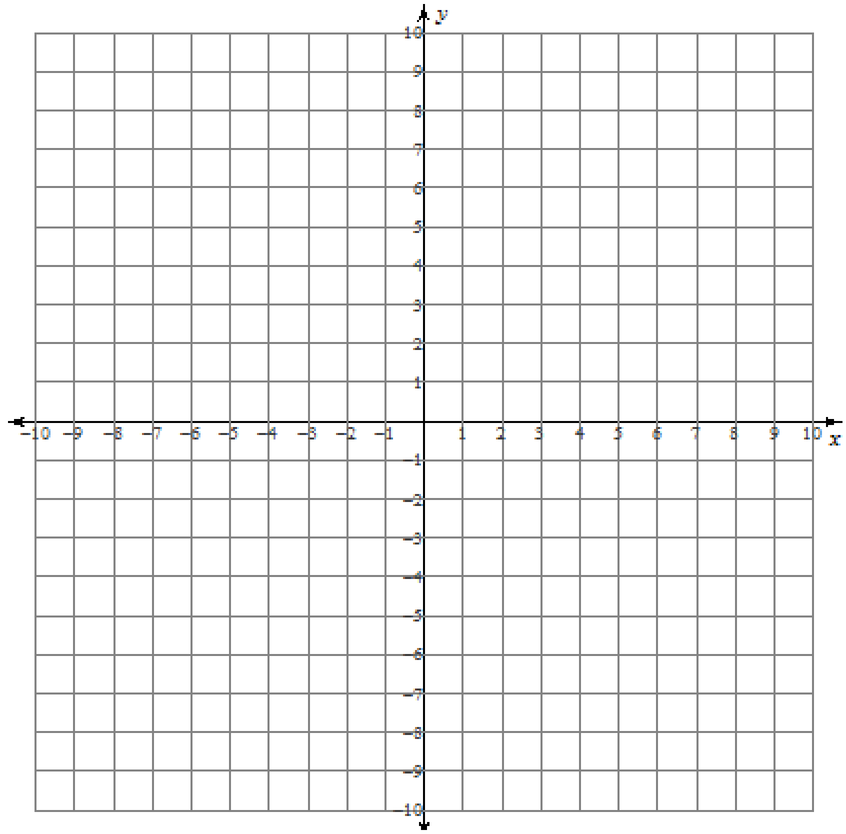
a. $f(x) = \dfrac{x^2 - 3x + 2}{x - 1}$

x-intercept(s)	
y-intercept	
Vertical Asymptote(s)	
Points of Discontinuity	
Horizontal Asymptote	
Other key points	
Domain	
Range	



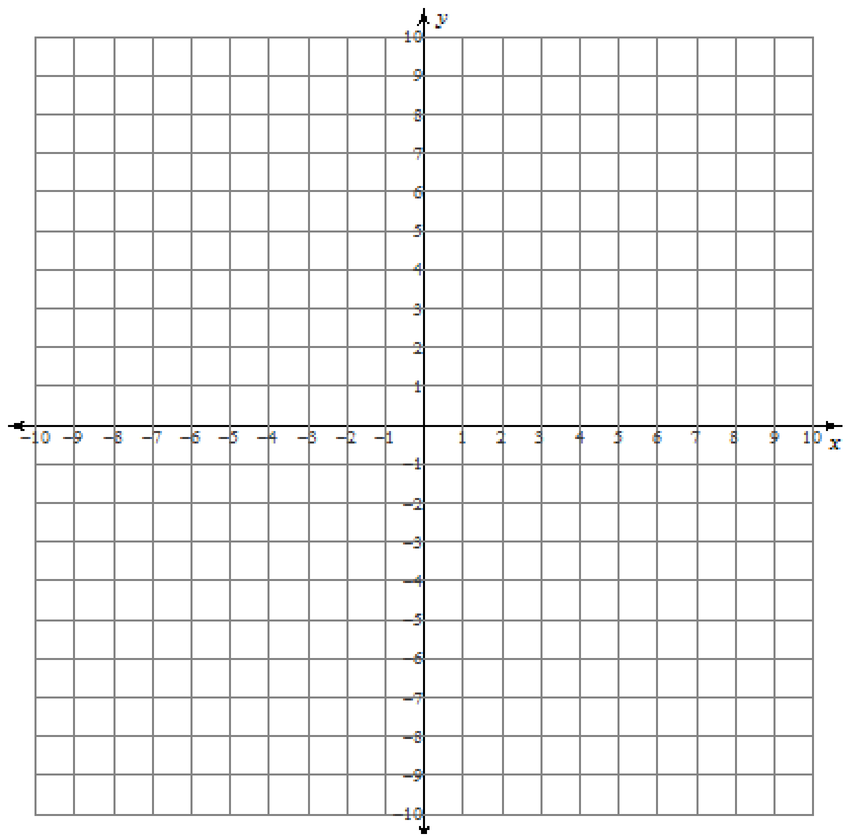
b. $f(x)=\dfrac{x-1}{x^2-x-2}$

x-intercept(s)	
y-intercept	
Vertical Asymptote(s)	
Point(s) of Discontinuity	
Horizontal Asymptote	
Other key points	
Domain	
Range	



c. $f(x)=\dfrac{x^2+2x-8}{x^2+5x+4}$

x-intercept(s)	
y-intercept	
Vertical Asymptote(s)	
Point(s) of Discontinuity	
Horizontal Asymptote	
Other key points	
Domain	
Range	

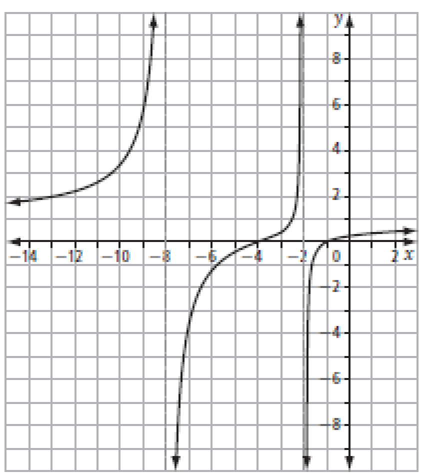


Example 3: Match Graphs and Equations for Rational Functions

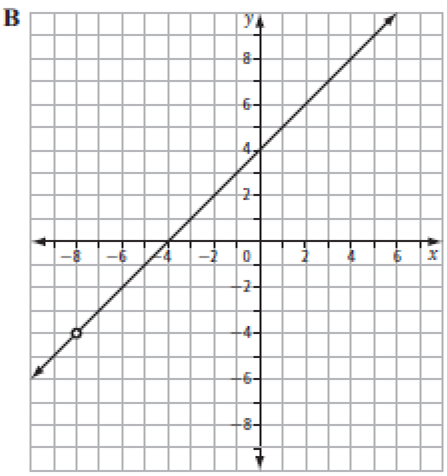
Match the equation of each rational function with its corresponding graph. Determine key features such as intercepts, asymptotes, and points of discontinuity to help you.

a. $f(x)=\dfrac{x+4}{x+8}$ b. $f(x)=\dfrac{x^2+12x+32}{x+8}$ c. $f(x)=\dfrac{x^2+12x+32}{x^2+10x+16}$ d. $f(x)=\dfrac{x^2+5x+4}{x^2+10x+16}$

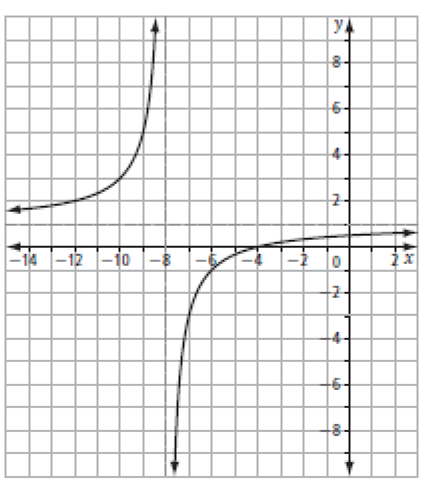
A



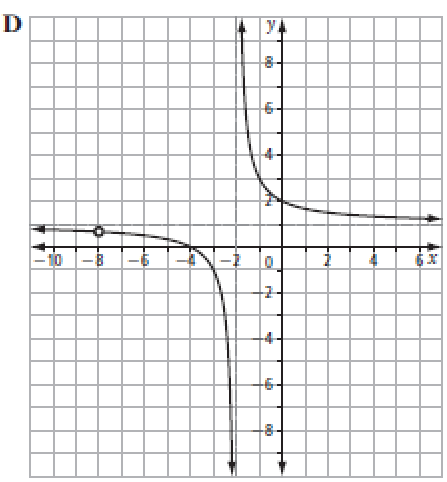
B



C



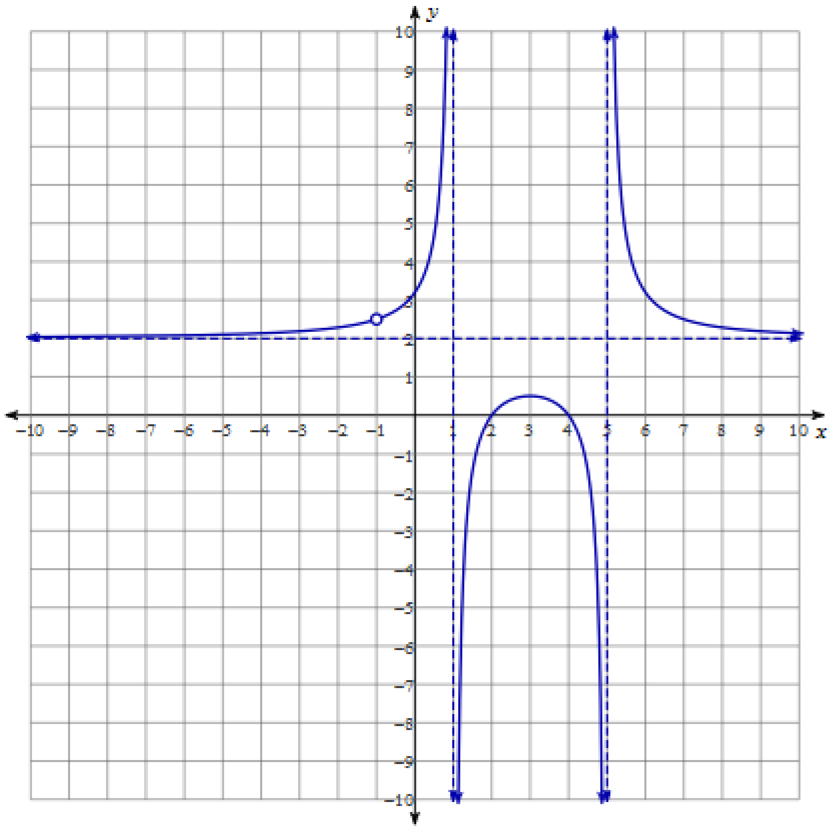
D



a. __________ b. __________ c. __________ d. __________

Example 4: Determining the Equation of a Rational Function

Determine the equation of the rational function illustrated below.



Solution:

Write the equation in factored form. Consider characteristics such as intercepts, asymptotes and points of discontinuity when determining the factors in the numerator and denominator.

Oblique Asymptotes

An oblique, or slant, asymptote is a line with non-zero slope that the graph of a function approaches as $x \to \pm\infty$.

Oblique asymptotes exist for rational functions in which the degree of the numerator is one more than that of the denominator, provided that the numerator is not entirely divisible by the denominator.

Determining the Equation of an Oblique Asymptote

Use long division to express a rational function, $f(x) = \dfrac{N(x)}{D(x)}$, as a *quotient* with a *remainder*. The quotient term will be a linear expression, $mx + b$. As x approaches $\pm\infty$, the remainder term will approach zero, so the graph of the function will approach the line $y = mx + b$.

Example 5: Investigating the Oblique Asymptote

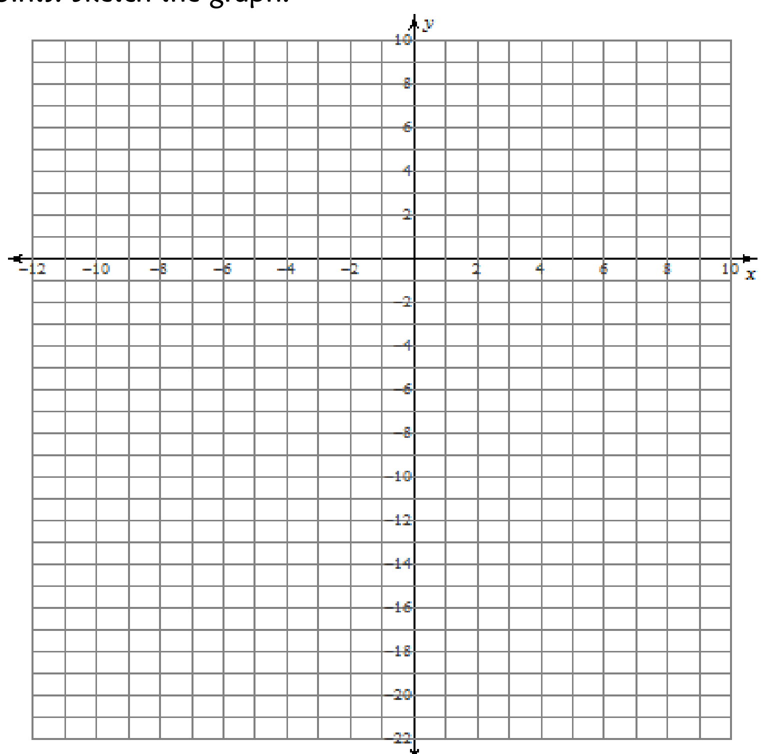
Sketch the graph of each function. a. $f(x) = \dfrac{x^2 - 3x}{x + 1}$ b. $f(x) = \dfrac{-x^3 - 1}{x^2 - 4}$

Solution:

Express the function in factored form. Determine any vertical, horizontal, and oblique asymptotes, points of discontinuity, intercepts, and any other necessary key points. Sketch the graph.

a. $f(x) = \dfrac{x^2 - 3x}{x + 1}$

x-intercept(s)	
y-intercept	
Vertical Asymptote(s)	
Point(s) of Discontinuity	
Horizontal Asymptote	
Oblique Asymptote	
Other key points	

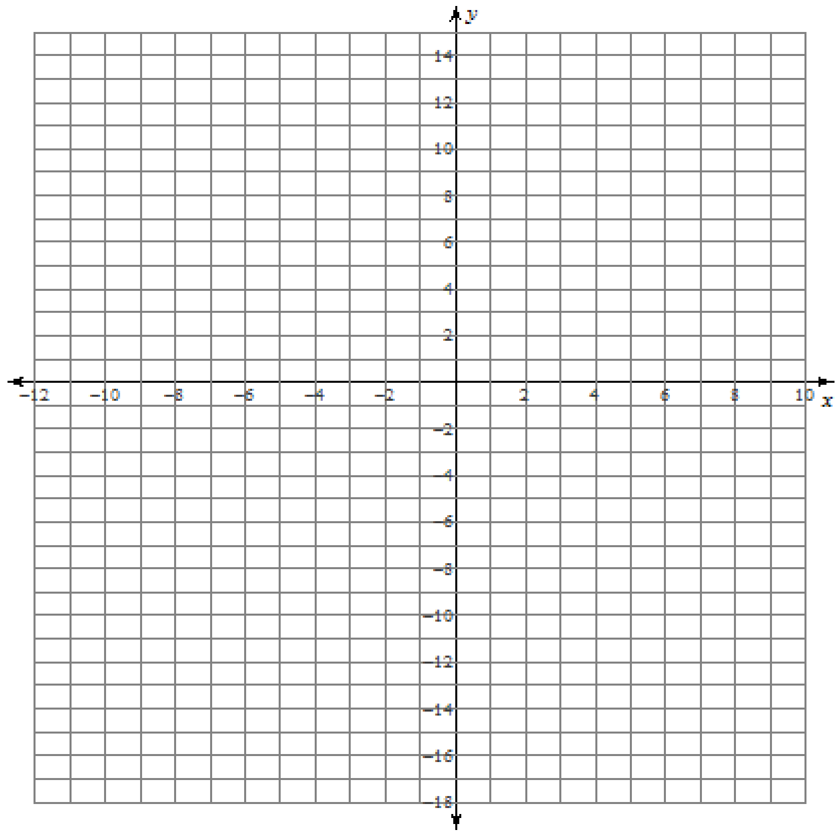


There is an oblique asymptote at $y = \lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \left(x - 4 + \dfrac{4}{x+1} \right) = x - 4$.

In other words, as x approaches $\pm\infty$, the *remainder* term ____________________ approaches ________, so the graph of *f(x)* approaches the line ________________________.

b. $f(x) = \dfrac{-x^3 - 1}{x^2 - 4}$

x-intercept(s)	
y-intercept	
Vertical Asymptote(s)	
Point(s) of Discontinuity	
Horizontal Asymptote	
Oblique Asymptote	
Other key points	

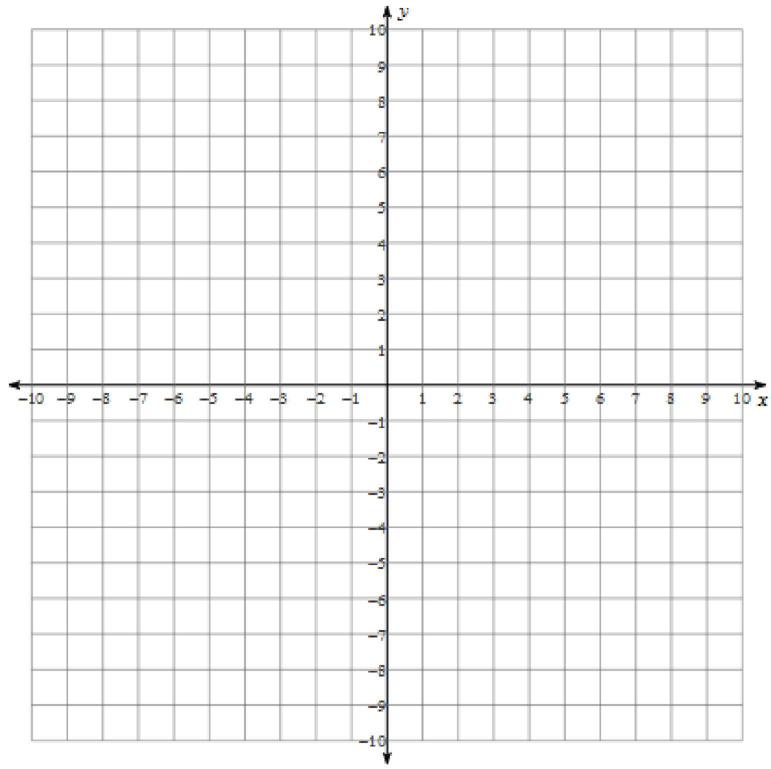


As x approaches $\pm\infty$, the *remainder* term ____________________ approaches ________, so the graph of *f(x)* approaches the line ________________________.

Example 6: Determining Where a Rational Function Crosses a Horizontal Asymptote

a. Sketch the graph of the function $f(x)=\dfrac{x^2+x-12}{x^2-4}$.

x-intercept(s)	
y-intercept	
Vertical Asymptote(s)	
Point(s) of Discontinuity	
Horizontal Asymptote	
Oblique Asymptote	
Coordinates of Point(s) where graph crosses the asymptote	
Other key points	

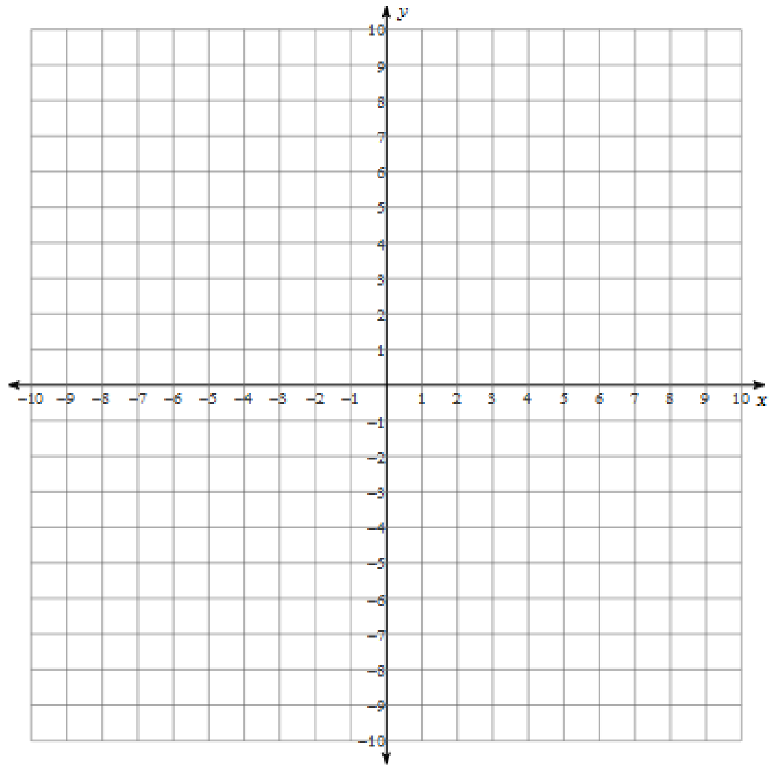


b. Determine where the function crosses the horizontal asymptote.

Example 7: Determining Where a Rational Function Crosses an Oblique Asymptote

a. Sketch the graph of the function $f(x)=\dfrac{x^3-3x^2+4}{x^2-2x+1}=\dfrac{(x-2)^2(x+1)}{(x-1)^2}$.

x-intercept(s)	
y-intercept	
Vertical Asymptote(s)	
Point(s) of Discontinuity	
Horizontal Asymptote	
Oblique Asymptote	
Coordinates of Point(s) where graph crosses the asymptote	
Other key points	

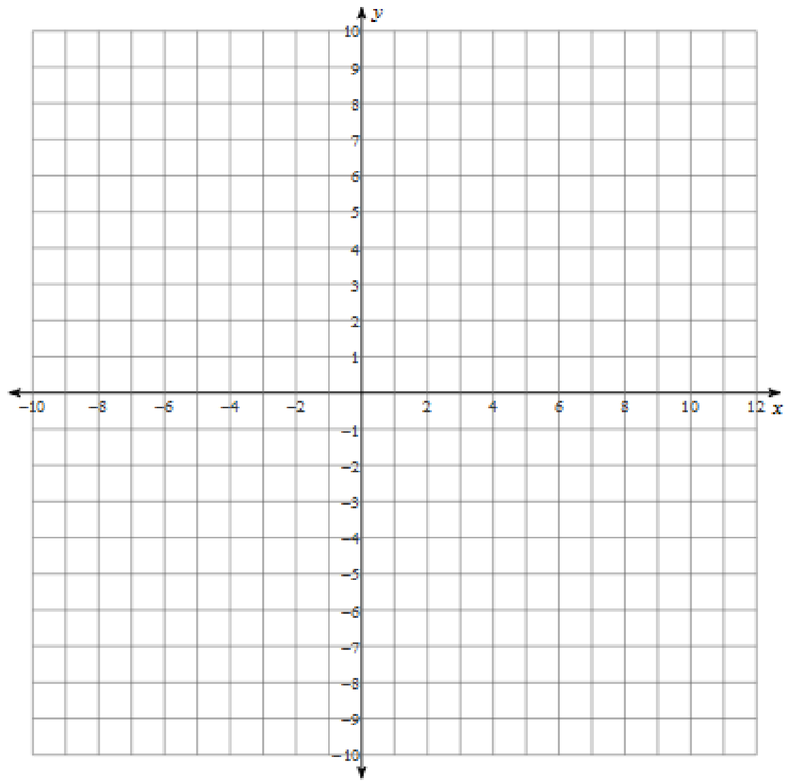


b. Determine where the function crosses the oblique asymptote.

Extra Practice: Crossing Asymptotes

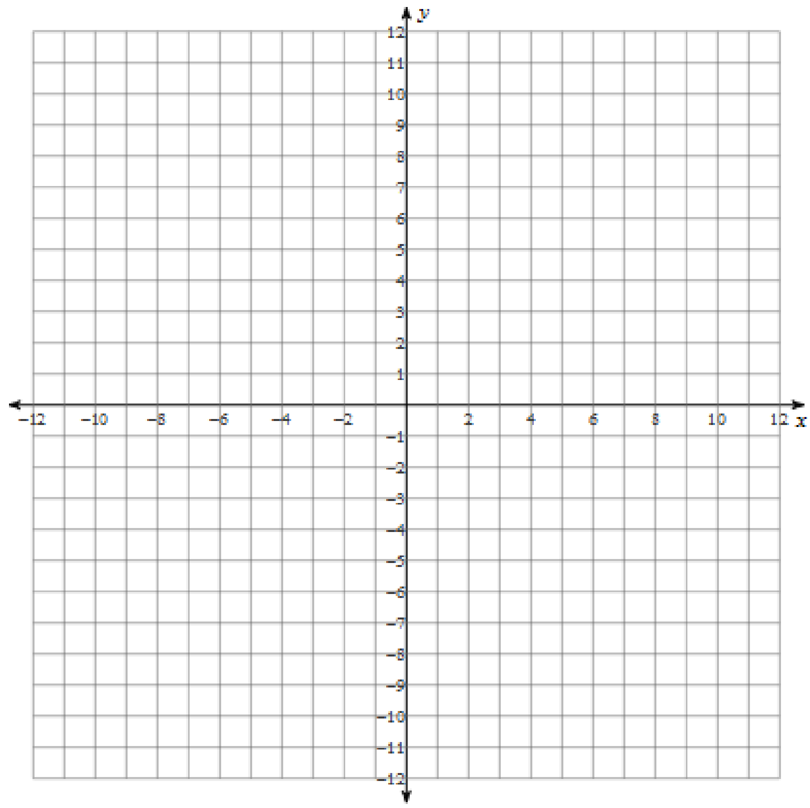
1. Sketch the graph of the function $f(x)=\dfrac{2x^3+6x^2-72x-216}{x^3-5x^2-8x+48}=\dfrac{2(x+3)(x+6)(x-6)}{(x+3)(x-4)^2}$ and determine if and where the function crosses the horizontal or oblique asymptote.

x-intercept(s)	
y-intercept	
Vertical Asymptote(s)	
Point(s) of Discontinuity	
Horizontal Asymptote	
Oblique Asymptote	
Coordinates of Point(s) where graph crosses the asymptote	
Other key points	



2. Sketch the graph of the function $f(x)=\dfrac{x^3+x^2-25x-25}{x^2-2x+1}=\dfrac{(x+1)(x+5)(x-5)}{(x-1)^2}$ and determine if and where the function crosses the horizontal or oblique asymptote.

x-intercept(s)	
y-intercept	
Vertical Asymptote(s)	
Point(s) of Discontinuity	
Horizontal Asymptote	
Oblique Asymptote	
Coordinates of Point(s) where graph crosses the asymptote	
Other key points	



3. Sketch the graph of the function $f(x)=\dfrac{x^4-x^3-12x^2-4x+16}{x^3-x^2-8x+12}=\dfrac{(x-1)(x-4)(x+2)^2}{(x+3)(x-2)^2}$ and determine if and where the function crosses the horizontal or oblique asymptote.

x-intercept(s)	
y-intercept	
Vertical asymptote(s)	
Point(s) of Discontinuity	
Horizontal asymptote	
Oblique asymptote	
Coordinates of Point(s) where graph crosses the asymptote	
Other key points