Continuity of a Function

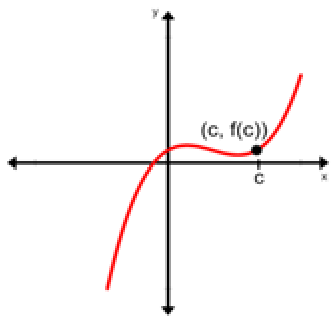
Continuity at a Point

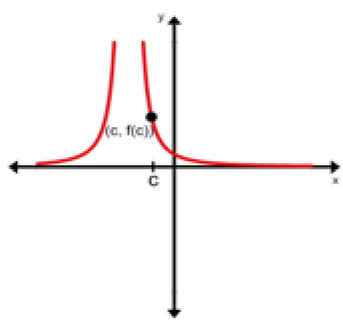
When we explored the limit of f(x) as x approaches c, the emphasis was on the function values *close* to x = c rather than what happens to the function *at* x = c. We will now consider the following cases.

- If $\lim_{x \to c} f(x) = f(c)$, then f(x) is *continuous* at x = c.
- If $\lim_{x \to c} f(x) \neq f(c)$, then f(x) is *discontinuous* at x = c.

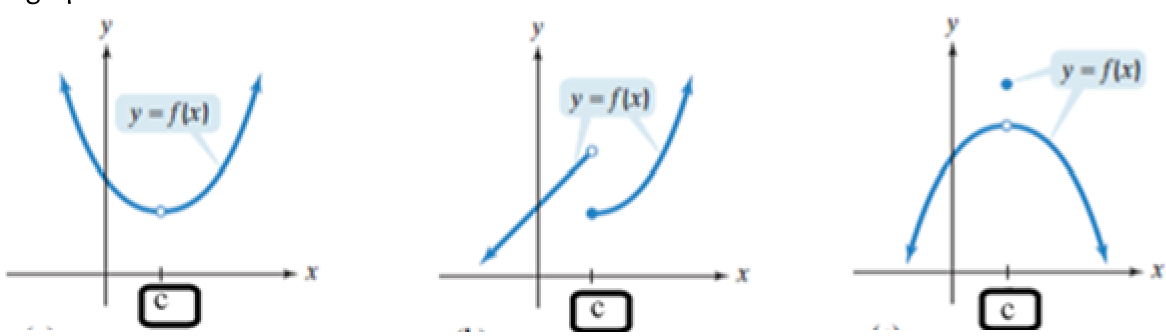
Graphically, the criterion for determining whether or not a function is continuous at x = c is to be able to draw the curve at, and near, x = c without lifting the pencil; there should be no holes or breaks.

Both graphs below illustrate functions that *are* continuous at x = c.



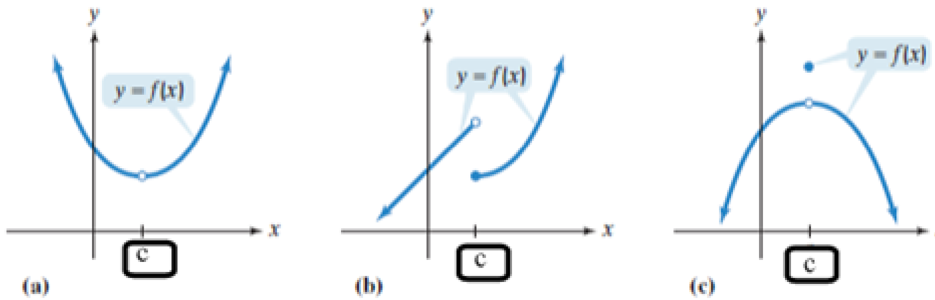


The three graphs below illustrate functions that are *not* continuous at x = c.



Three Tests for Continuity at a Point
1. $f(c)$ is defined
2. $\lim_{x \to c} f(x)$ exists; the left- and right-hand limits are equal
3. $\lim_{x \to c} f(x) = f(c)$
If any of these three tests fail, the function is discontinuous at x = c.

Let's take another look at the graphs below to see why these functions are not continuous at x = c, taking the tests for continuity into account.

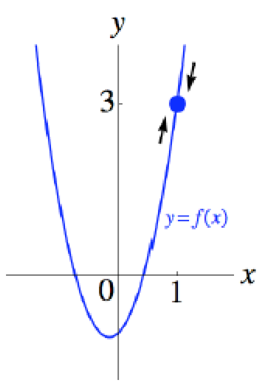


a. In graph (a), the open dot indicates that there is no point on the graph corresponding to x = c. This means that f(c) is not defined. Although the limit as x approaches c exists, *condition 1* is not satisfied, therefore, the function is not continuous.

b. In graph (b), the closed dot at x = c shows that f(c) is defined. However, the limit of the function as x approaches c from the left is *not* equal to the limit of the function as x approaches c from the right, therefore, $\lim_{x \to c} f(x)$ does not exist. *Condition 2* is not satisfied, therefore, the function is not continuous.

c. In graph(c), the closed dot at x = c shows that f(c) is defined. Also, $\lim_{x \to c} f(x)$ exists. However, there is still an interruption at c. This is because $\lim_{x \to c} f(x) \neq f(c)$. *Condition 3* is not satisfied, therefore, the function is not continuous.

By contrast, the function $f(x) = 3x^2 + x - 1$ is continuous at, say, x = 1 since all three conditions of continuity are satisfied.



1. $f(1) = 3(1)^2 + 1 - 1 = 3$
2. $\lim_{x \to 1}(3x^2 + x - 1) = 3$
3. $\lim_{x \to 1} f(x) = f(1) = 3$

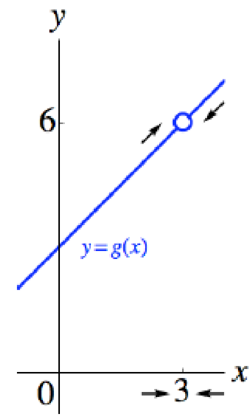
Types of Discontinuity

The following examples show four different types of discontinuity: holes, jumps, breaks, and poles. These fall into the following two categories.

Removable discontinuity: You can make the function continuous by defining or changing the function value f(c).

Non-removable discontinuity: You cannot make the function continuous by any value f(c).

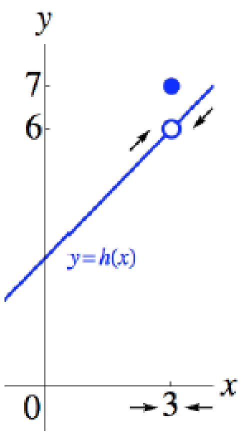
Example 1:



In the function illustrated to the right, there is a *hole discontinuity* at (3, 6).
If this point were defined (ie. g(3)=6), the function would be continuous.

Therefore, at x = 3, there is a *removable discontinuity.*

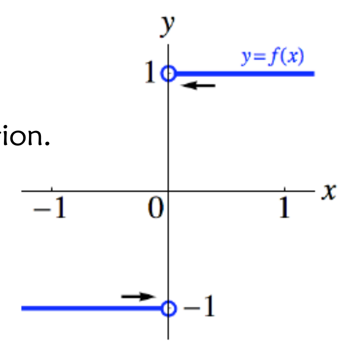
Example 2:



In the function illustrated to the left, there is a *jump discontinuity* at x = 3.
If this jump value were changed to fill the hole at (3, 6), the function would be continuous.

Therefore, at x = 3, there is a *removable discontinuity.*

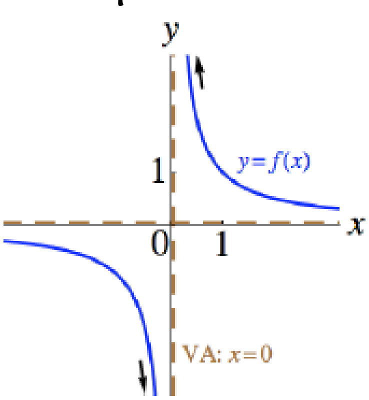
Example 3:



In the function illustrated to the right, there is a *break discontinuity* at x = 0.
There is no value for f(x) that would connect this break and create a continuous function.

Therefore, at x = 0, there is a *non-removable discontinuity.*

Example 4:



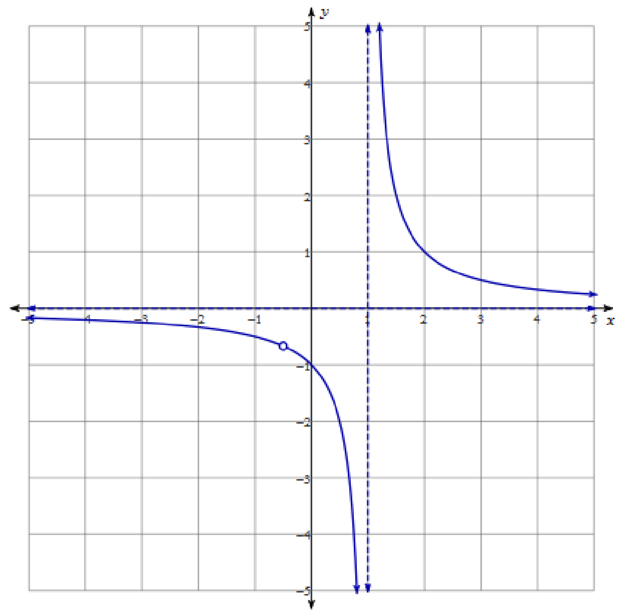
In the function illustrated to the left, there is a vertical asymptote at x = 0.
This is referred to as a *pole* or *infinite discontinuity.*
No defined value of f(x) would make this function continuous.

Therefore, at x = 0, there is a *non-removable discontinuity.*

Example 5: Determining Continuity at a Point

Determine whether the function $f(x)=\dfrac{2x+1}{2x^2-x-1}$ is continuous at the given points.

a. x = 2 b. x = 1



Solution:

a. Use the three tests for continuity at a point.

1. Determine if $f(2)$ is defined.

$f(2)=$

2. Determine if $\lim\limits_{x\to 2} f(x)$ exists.

$$\lim_{x\to 2}\left(\frac{2x+1}{2x^2-x-1}\right)=$$

3. Determine if $\lim\limits_{x\to 2} f(x)=f(2)$

$f(2)=1$ and $\lim\limits_{x\to 2} f(x)=1$

$\therefore\ \lim\limits_{x\to 2} f(x)=f(2)$

All 3 conditions are satisfied, so f(x) is continuous at x = 2.

b. Use the three tests for continuity at a point.

1. Determine if $f(1)$ is defined.

$f(1)=$

Because f(1) is undefined, it is not necessary to proceed with the other tests.

f(x) is *not* continuous at x = 1.

Note: There is a *pole* or *infinite* (ie. *non-removable*) discontinuity at x = 1.

Example 6: Determining where a Function is Discontinuous

For what values of x, if any, is the function $f(x)=\dfrac{x^2-4}{x^2+3x+2}$ discontinuous?

Solution:

Since f(x) is a rational function, the function will be discontinuous at values of x for which the function is undefined. We can find these values by determining the zeroes of the denominator.

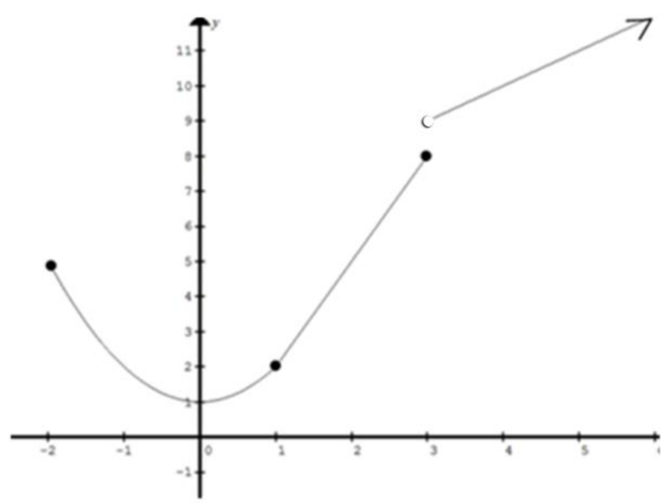
$x^2+3x+2=0$

Therefore, f(x) is discontinuous at x = ______ and x = ______.

Example 7: Determining Continuity at a Point

Determine if the function $f(x)=\begin{cases} x^2+1, & -2<x\le 1 \\ 3x-1, & 1<x\le 3 \\ x+6, & x>3 \end{cases}$ is continuous at x = 3.

Solution:



Use the three tests for continuity at a point.

1. Determine if $f(3)$ is defined.

   $f(3)=$

2. Determine if $\lim\limits_{x\to 3} f(x)$ exists.

   $\lim\limits_{x\to 3^-} f(x)=$

   $\lim\limits_{x\to 3^+} f(x)=$

   Since $\lim\limits_{x\to 3^-} f(x)\ne \lim\limits_{x\to 3^+} f(x)$ then $\lim\limits_{x\to 3} f(x)$ DNE. It is not necessary to proceed with the third test.

f(x) is *not* continuous at x = 3.

Note: There is a *break* (ie. *non-removable*) discontinuity at x = 3.

Example 8: Determining Continuity at a Point

Determine if the function $f(x)=\begin{Bmatrix} |x+2|, & x\le -6 \\ x^2-32, & x>-6 \end{Bmatrix}$ is continuous at x = –6.

Solution:

Use the three tests for continuity at a point.

1. Determine if $f(-6)$ is defined.

   $f(-6)=$

2. Determine if $\lim\limits_{x\to -6} f(x)$ exists.

   $\lim\limits_{x\to -6^-} f(x)=$

   $\lim\limits_{x\to -6^+} f(x)=$

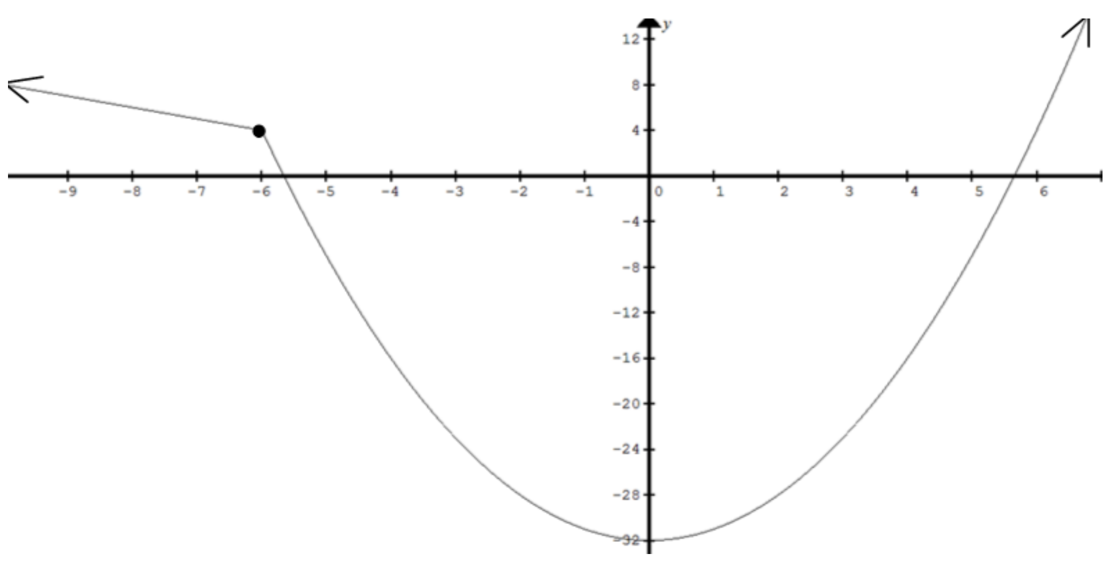
   Since $\lim\limits_{x\to -6^-} f(x)=\lim\limits_{x\to -6^+} f(x)=4$, then $\lim\limits_{x\to -6} f(x) = 4$.

3. Determine if $\lim\limits_{x\to -6} f(x)=f(-6)$

   $f(-6)=4$ and $\lim\limits_{x\to -6} f(x)=4$

   $\therefore \lim\limits_{x\to -6} f(x)=f(-6)$

All 3 conditions are satisfied, so f(x) is continuous at x = –6.



Example 9: Determining Continuity at a Point

Find the constant k such that $f(x)=\begin{cases} kx, & x\le 2 \\ x^2-4x+3, & x>2 \end{cases}$ is a continuous function at x = 2.

Solution:

1. Determine if f(2) is defined.

   f(2) =

   Although we do not yet know the value of 2k, we do know that k is a constant, therefore, 2k is defined.

2. Determine the value of k such that $\lim_{x\to 2} f(x)$ exists.

   $\lim_{x\to 2^-} f(x)=$ $\lim_{x\to 2^+} f(x)=$

   $\lim_{x\to 2} f(x)$ exists if $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)$

   So, 2k = –1 → k = ________

3. Verify that if k = –½, then $\lim_{x\to 2} f(x)=f(2)$

   $\lim_{x\to 2} f(x) = -1$

   f(2) = 2(–½) = –1

   $\therefore \lim_{x\to 2} f(x)=f(2)$

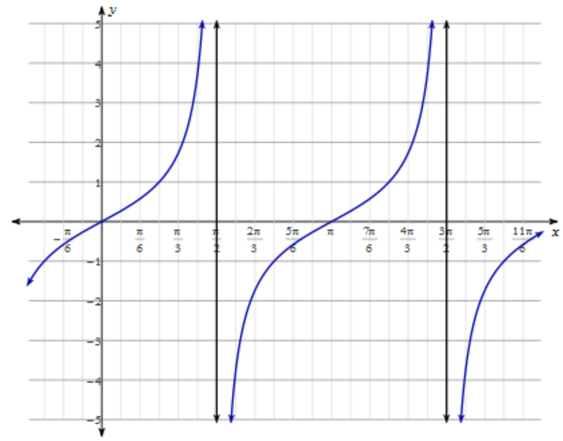
Therefore, if k = –½, then f(x) is continuous at x = 2.

Continuity on an Interval

In previous examples, we looked at continuity of a function at a given point. We will now look at continuity of a function on intervals. Many functions, such as polynomial and trigonometric functions, are continuous at every point in their domains. When describing the continuity of a function, it is therefore useful to first identify the function's domain. We can then describe the continuity of the function along intervals within the domain.

Continuity on an Open Interval

An open interval is a continuous set of real numbers that does not contain its endpoints. A function is continuous on an open interval (a, b) if it is continuous at every point in (a, b). It does *not* have to be continuous at the endpoints.



Example 10: Describing the Continuity of a Function

Describe the continuity of the function f(x) = tan x.

Step 1) Identify the domain.

The domain is the set of all real numbers, $x \neq \frac{\pi}{2} + \pi n,\ n \in Z$

Step 2) Identify the intervals in the domain.

The function is discontinuous at each asymptote, x = …, $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$.
It has open intervals between the asymptotes.

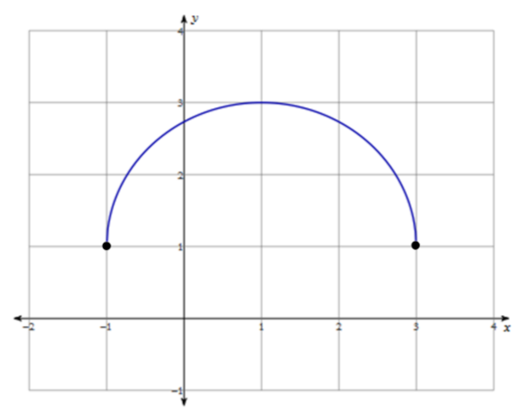
Step 3) Determine continuity on the intervals.

The function is continuous at all points other than the asymptotes.
It is therefore continuous on the open intervals between the asymptotes.

Continuity on a Closed Interval

A closed interval is a continuous set of real numbers that contains its endpoints. A function is continuous on a closed interval [a, b] if:

i. it is continuous at every point in (a, b).
ii. $\lim_{x \to a^+} f(x) = f(a)$ (ie. f(x) is continuous *from the right* at x = a.)
iii. $\lim_{x \to b^-} f(x) = f(b)$ (ie. f(x) is continuous *from the left* at x = b.)



Example 11: Describing the Continuity of a Function

Describe the continuity of the function $f(x) = 1 + \sqrt{-x^2 + 2x + 3}$.

Step 1) Identify the domain.

The domain is $x \in [-1,\ 3]$

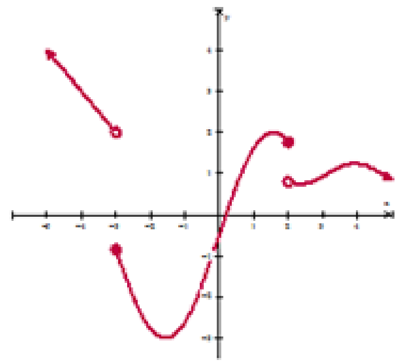
Step 2) Determine continuity on the interval and at each endpoint.

The function is continuous at each point on the open interval (–1, 3).

$\lim_{x \to -1^+} f(x) = f(-1) =$ _____ (ie. f(x) is continuous from the right at x = –1)
$\lim_{x \to 3^-} f(x) = f(3) =$ _____ (ie. f(x) is continuous from the left at x = 3)

Since f(x) is continuous on the open interval and at each endpoint, it is continuous on the closed interval $[-1,\ 3]$

Example 12: Where is a Function Continuous?



Where is the function, illustrated to the right, continuous?

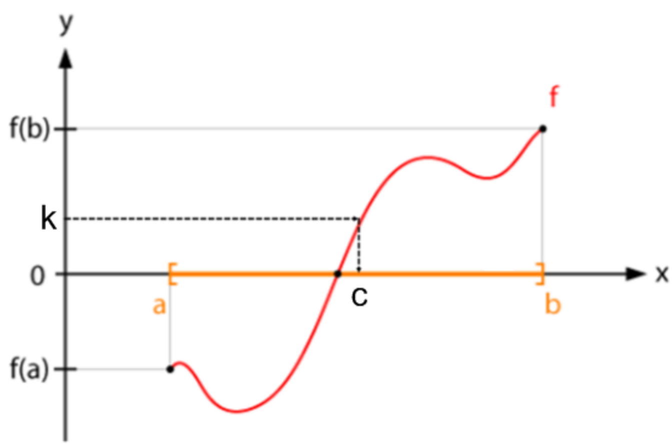
Solution:

The function shown is discontinuous at x = ______ and x = ______.

It is continuous on each of the intervals ______________, _______________, and _______________.

Intermediate Value Theorem

The figure below shows a continuous function f on the interval [a,b] and a number k that is between f(a) and f(b).



Since f is continuous on the interval [a, b], its graph can be drawn from (a, f(a)) to (b, f(b)) without lifting the pencil from the paper.

As the graph indicates, there is no way to do this unless the function crosses the horizontal line at y = k at least once between x = a and x = b. The coordinates of a point where this happens is (c, f(c)), or (c, k).

This idea leads to the intermediate value theorem.

> **Intermediate Value Theorem**
>
> If f is continuous on a closed interval [a, b] and k is any number between f(a) and f(b) inclusive, then there is at least one number c in the interval [a, b] such that f(c)=k.

Notice in the graph above that the curve extends *below* f(a). The Intermediate Value Theorem only says that an x-value for any y-value *between* f(a) and f(b) must exist. Also, notice that the theorem does *not* say that only *one* x-value exists for a certain k. In the graph above, *show an example of a value of k for which there are three corresponding values of x.*

Most people take this theorem for granted in some common situations.

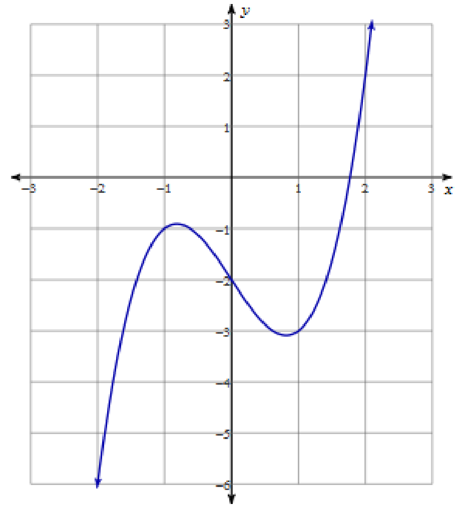
- If you dove to pick up a shell 15 feet below the surface of a lagoon, then at some instant in time you were 12 ft below the surface. You cannot go from the surface to 15 ft below without passing 12 ft.

- If you started driving from a stop (velocity =0 km/h) and accelerated to a velocity of 30 km/h, then there was an instant when your velocity was exactly 10 km/h.

The intermediate value theorem cannot be applied if the function is not continuous.

- Prices, taxes, & rates of pay change in jumps, or discrete steps, without taking on the intermediate values.

Example 13: The Intermediate Value Theorem and Zeros



Show that $f(x) = x^3 - 2x - 2$ has a zero in the interval [0, 2].

Solution:

Step 1) Calculate f(x) at the left endpoint of the interval.

f(0) =

Step 2) Calculate f(x) at the right endpoint of the interval.

f(2) =

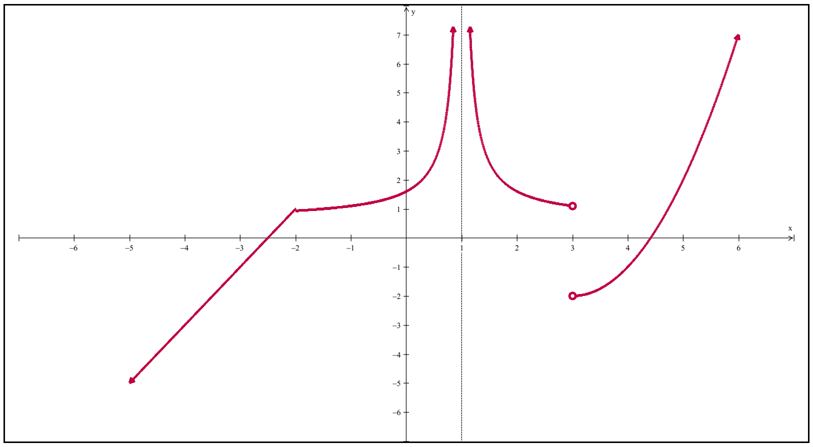
Step 3) Apply the Intermediate Value Theorem.

A zero of a function exists when f(x) = 0. Since this y-value is between f(0) = –2 and f(2) = 2, then there is at least one value c in the interval [0, 2] such that f(c) = 0.

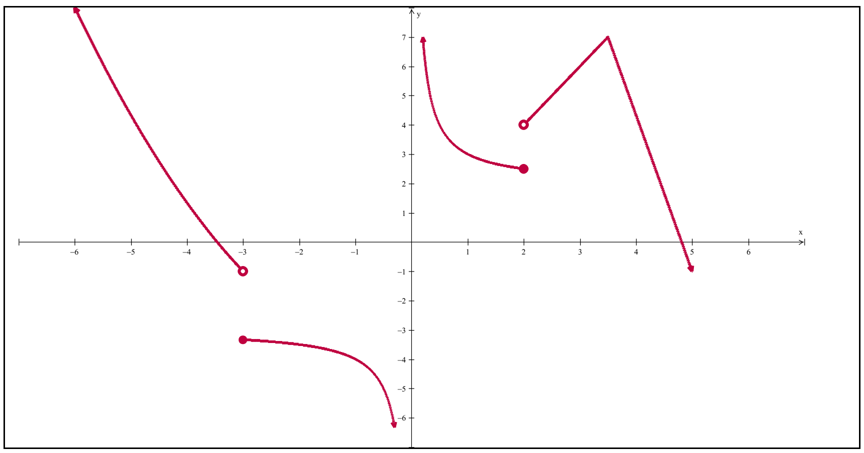
Note:
The Intermediate Value Theorem is an *existence theorem*. We can use it to determine the existence of a value c, but the theorem does not help us determine what that value is.

Practice:

1. Given the graph of the function f, find all the values in the domain of f at which f is **not** continuous.



2. Find all values in the function below where it is defined but **not** continuous.



In questions 3 to 5, use the *definition of continuity at a point* to show that the function is continuous at the given x-value.

3. $f(x) = x^2 + 3x + 5$ at $x = 3$

4. $g(x) = \frac{x+3}{(x^2-x-1)(x^2+1)}$ at $x = -2$

5. $h(x) = \frac{x\sqrt{x}}{(x-4)^2}$ at $x = 16$

Explain why the functions in questions 6 to 8 are *not* continuous at the given x-values.

6. $f(x) = \frac{1}{(x-3)^3}$ at $x = 3$

7. $g(x) = \frac{(x^2+4)}{(x^2-x-2)}$ at $x = 2$

8. $h(x) = \frac{x^2+4x+3}{x^2-x-2}$ at $x = -1$

9. Algebraically determine whether or not the following function is continuous at x = –1 and/or at x = 3. Provide an explanation as to why the function is or is not continuous.

$$f(x) = \begin{cases} \frac{x^3+3}{2}, & x \le -1 \\ -4x-3, & -1 < x \le 3 \\ \sqrt{7x+4}+10, & x > 3 \end{cases}$$

10. For what value of b is the following function continuous at $x = 3$?

$$f(x) = \begin{cases} bx+4 & if\ x \le 3 \\ bx^2-2 & if\ x > 3 \end{cases}$$

Answers:

1) $f(x)$ is *not continuous* at $x = 1$ and $x = 3$

2) $f(x)$ is *defined* but *not continuous* at $x = -3$ and at $x = 2$.

Note: At $x = 0$, f(x) is not continuous but also is *not defined*.

3) 1. $f(3) = (3)^2 + 3(3) + 5 = 23$ $\rightarrow$ The function f(x) is defined at x = 3.
2. $\lim_{x \to 3} f(x) = 23$ $\rightarrow$ The limit of f(x) exists as x approaches 3.
3. $f(3) = \lim_{x \to 3} f(x)$

The three conditions of continuity are satisfied, therefore, the function f(x) is continuous at x = 3.

4) 1. $g(-2) = \frac{-2+3}{\left((-2)^2-(-2)-1\right)\left((-2)^2+1\right)} = \frac{1}{(5)(5)} = \frac{1}{25}$ $\rightarrow$ The function g(x) is defined at x = -2.
2. $\lim_{x \to -2} g(x) = \frac{1}{25}$ $\rightarrow$ The limit of g(x) exists as x approaches -2.
3. $g(-2) = \lim_{x \to -2} g(x)$

The three conditions of continuity are satisfied, therefore, the function g(x) is continuous at x = -2.

5) 1. $h(16) = \frac{16\sqrt{16}}{(16-4)^2} = \frac{64}{(12)^2} = \frac{64}{144} = \frac{4}{9}$ $\rightarrow$ The function h(x) is defined at x = 16.
2. $\lim_{x \to 16} h(x) = \frac{4}{9}$ $\rightarrow$ The limit of h(x) exists as x approaches 16.
3. $h(16) = \lim_{x \to 16} g(x)$

The three conditions of continuity are satisfied, therefore, the function h(x) is continuous at x = 16.

6) $f(x)$ is not continuous at $x = 3$ since $f(3)$ is undefined.

7) $g(x)$ is not continuous at $x = 2$ since $g(2)$ is undefined.

8) $h(x)$ is not continuous at $x = -1$ since $h(-1)$ is undefined.

9) At $x = -1$:

1. $f(-1) = \dfrac{(-1)^3 + 3}{2} = 1$
2. $\lim\limits_{x \to -1^-} f(x) = 1$

   $\lim\limits_{x \to -1^+} f(x) = 1$

   $\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^+} f(x), \ \therefore \lim\limits_{x \to -1} f(x) = 1$
3. $f(-1) = \lim\limits_{x \to -1} f(x)$

The three conditions of continuity are satisfied, so f(x) is continuous at x = -1.

At $x = 3$:

1. $f(3) = -4(3) - 3 = -15$
2. $\lim\limits_{x \to 3^-} f(x) = -15$

   $\lim\limits_{x \to 3^+} f(x) = 15$

   $\lim\limits_{x \to 3^-} f(x) \neq \lim\limits_{x \to 3^+} f(x), \ \therefore \lim\limits_{x \to 3} f(x)$ DNE

   So f(x) is not continuous at x = 3

10) If f(x) is continuous at x = 3, then:

1. $f(3)$ must be defined $\to f(3) = 3b + 4$
2. $\lim\limits_{x \to 3} f(x)$ must exist, which means that $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^+} f(x)$.

   So, determine the left and right-hand limits, set them equal, and determine the value of b:

   $\lim\limits_{x \to 3^-} f(x) = 3b + 4$

   $\lim\limits_{x \to 3^+} f(x) = 9b - 2$

   $3b + 4 = 9b - 2 \ \to \ 6 = 6b \ \to \ b = 1$

   So $\lim\limits_{x \to 3} f(x) = 7$ when $b = 1$
3. $f(3) = \lim\limits_{x \to 3} f(x) = 7$

$\therefore \ f(x)$ is continuous at $x = 3$ if $b = 1$